International Journal of Instruction
e-ISSN: 1308-1470 ● www.e-iji.net

April 2024 ● Vol.17, No.2
p-ISSN: 1694-609X
pp. 303-320

Article submission code: 20230321222524
Received: 21/03/2023
Revision: 18/09/2023
Accepted: 07/10/2023
OnlineFirst: 11/01/2024

Components and Subcomponents of the Slope used by High School Mathematics Teachers

Martha Iris Rivera López
Autonomous University of Guerrero, Mexico, *irivera@uagro.mx*

Lourdes Soto Velázquez
Autonomous University of Guerrero, Mexico, *10224827@uagro.mx*

Gustavo Martínez Sierra
Autonomous University of Guerrero, Mexico, *gmartinezsierra@gmail.com*

The present study aims to identify and describe the components and subcomponents of the slope used by a group of seven high school mathematics teachers to solve tasks involving it. Its importance is due to the unquestionable role of the teacher as promoter of the learning of his students, since he considers and transforms his knowledge about a content to put it into play in his teaching practice. Data were collected through a Task-Based Interview, whose protocol consisted of six tasks (two situations with real-life context and four questions associated with teaching and learning the concept). Data analysis was code-by-code using the statements and procedures used by the teachers and the framework of the components and subcomponents of the slope. The findings of this exploratory study showed that teachers have conceptualized the concept in the Trigonometric Conception and Constant Ratio components, mainly in the visual-procedural and non-visual-procedural approaches. On the other hand, their interpretation of the concept makes it difficult to establish its relationship with real-world situations that allude to variational ideas. These results suggest the implementation of professional development courses to broaden their understanding and interpretation of the slope in authentic and realistic scenarios involving geometric and variational notions.

Keywords: slope, mathematics teachers, high school, task-based interview, learning

INTRODUCTION

The concept of slope has several ways of being conceptualized, each of which holds fundamental importance across multiple domains (Cho & Nagle, 2017; Nagle et al., 2022; Stump, 1999). These conceptualizations can be categorized into two notions: a geometric notion and a variational notion. In the geometric sense, as a measure of inclination (Lobato & Thanheiser, 2002), it is often associated with physical situations (e.g., ramps, stairs, slides, mountains). In the variational sense, as a measure of the change of one variable with respect to another (rate of change), it enables the

Citation: López, M. I. R., Velázquez, L. S., & Sierra, G. M. (2024). Components and subcomponents of the slope used by high school mathematics teachers. *International Journal of Instruction*, *17*(2), 303-320. https://doi.org/10.29333/iji.2024.17217a

differentiation of relationships between linear and nonlinear functions (Nagle et al., 2022; Teuscher & Reys, 2010; 2012). This concept has significant applications both in school mathematics and in everyday life, underscoring its significance as a foundational concept to be taught from elementary education to higher education (Dolores et al., 2020). Nevertheless, the literature Mathematics Education reports that students encounter challenges and make errors when approaching it as both a rate of change and a measure of inclination. For instance, they struggle with interpreting functional scenarios, such as the law of demand, current intensity, velocity, acceleration, static friction, and kinetic friction (Dolores et al., 2019; Hoban, 2021).

In that same sense, regarding both physical and functional situations, evidence has been found that students do not transfer their knowledge of slope between different types of problems. For instance, from a mathematical context to one involving real-world environments, or vice versa (Hoban, 2021; Nagle & Moore-Russo, 2013a; Rivera & Dolores, 2019; Stump, 1999, 2001). However, this may be a consequence of the instruction received, since the teacher's knowledge is a fundamental basis and plays an important role in the development of his teaching practice, as well as exerting a strong influence on what their students learn (Ismail & Jarrah, 2019; Sharp et al., 2019).

Despite the significance of the teacher's role, there has been limited inquiry into their knowledge of the slope concept and its integration within their instructional practice. In this context, Paolucci and Stepp (2021) observed that prospective American teachers' knowledge for designing tasks to teach the concept of slope tends to emphasize real-world physical situations, while showing limited awareness of contexts that promote functional situations. Moreover, they also highlighted a difficulty in considering variables other than distance and time. On the other hand, Nagle et al. (2013a) found that American university teachers conceptualize slope in various ways, viewing it as a functional property (a constant rate of change), a physical property (referring to the inclination of a line), a geometric ratio (graphically represented as rise over run), and an algebraic ratio (formulated as $((y_2\text{-}y_1) / (x_2\text{-}x_1))$). This interpretation, as partially aligned with the findings reported by Stanton and Moore-Russo (2012), corresponds, to some extent, with the expectations of the American curriculum spanning grades 1 through 12.

In Mexico, Salgado and Dolores (2021) reported that high school Mexican teachers conceptualize slope primarily as an algebraic ratio and as a physical property among the majority of their participants, while conceptualizations associated with variational ideas are the least likely they refer. However, Dolores et al. (2020) identified that the Mexican curriculum suggests promoting the treatment of the concept through variational aspects and real-world situations from elementary education to high school level. Hence, we propose the idea of reflecting on whether the teacher addresses what the curriculum demands, if their knowledge allows for it, even more so if they can transfer their knowledge to real-life situations.

Therefore, the objective of this study is to answer the following research question: Which slope components and subcomponents (visual, non-visual, procedural, conceptual) do high school level teachers emphasize when solving tasks involving it?

Identifying the components and subcomponents of slope utilized by teachers in various scenarios will offer insight into the network of connections present in their knowledge. Furthermore, this process will help pinpoint research gaps and establish guidelines for designing a professional development course for teaching and learning the concept of slope. This design will involve creating connections among the five main components of the concept, as well as their corresponding subcomponents.

Components and Subcomponents of the Slope

The various representations of slope were categorized and termed as slope conceptualizations by Moore-Russo et al. (2011). While diverse researchers have employed this framework to explore those commonly used by students, teachers, those demanded by curricula, and even those promoted by textbooks, the results have largely focused on providing isolated descriptions of what and how these appear, rather than considering relationships among them (e.g., Dolores et al., 2020; Dolores & Mosquera, 2022; Hoffman, 2015; Nagle & Moore-Russo, 2013a; Nagle et al. 2013; Salgado & Dolores, 2021; among others). In response to this, Nagle and Moore-Russo (2013b) proposed a network of five components to establish connections among these conceptualizations by combining procedural and conceptual comprehension approaches with visual and analytical interpretations, thereby facilitating the linkage of both their geometric and variational notions, resulting in a more robust understanding of the concept (see Table 1).

Table 1
Slope components and subcomponents (adapted from Nagle & Moore-Russo, 2013b)

Slope component	Description	Subcomponents (indicated with subscripts) v= visual, n= non-visual, p= procedural, c= conceptual	
Constant Ratio (R)	Slope viewed as a ratio in visual (rise/run) or analytic (change in *y* over change in *x*) form; it helps explain why linear behavior results in a constant ratio.	$R_{v,p}$: It is used as rise/ run or vertical change/horizontal change in the graph of a line.	$R_{v,c}$: Similarity of slope triangles yields a constant ratio of rise/run regardless of the position on the graph.
		$R_{n,p}$: It is used as change in *y* over change in *x*, that is $\Delta y/\Delta x$, or use $(y_2-y_1)/(x_2-x_1)$.	$R_{n,c}$: Recognition of slope as a constant rate of change between two covarying quantities; an equivalence class of ratios and hence a function.
Behavior Indicator (B)	Relates slope to the increasing or decreasing behavior of a linear function or graph; links sign of the value *m* with the function or graph's behavior	$B_{v,p}$: Increasing (or decreasing) lines have a positive (or negative) slope, horizontal lines have zero slope.	$B_{v,c}$: The positive rise corresponds to a positive run for an increasing line, yielding a positive slope. For a decreasing line, a negative rise corresponds to a positive run, yielding a negative slope.
		$B_{n,p}$: The value of *m* in the equation of a linear function (e.g., in $y=mx+b$) indicates whether *f* is an increasing ($m>0$), decreasing ($m<0$), or constant ($m=0$) linear function.	$B_{n,c}$: Application of the definition, if the function *f* is increasing then $f(x_1)<f(x_2)$ for each $x_1<x_2$, then $[f(x_2)-f(x_1)]/(x_2-x_1)>0$; Similar generalization for the decreasing function.
Trigonometric Conception (T)	Describes slope in terms of the angle of inclination of a line with a horizontal; it extends to relates steepness to the determination of tangent of the angle of inclination.	$T_{v,p}$: Steepness of a line, slope as the angle of inclination of the line with a horizontal; as a line is rotated about a point, the slope changes.	$T_{v,c}$: The angle of inclination determines the rise/run; a steeper line has a greater rise per given run than a less steep line.
		$T_{n,p}$: Slope is calculated as $tan\ \theta$, where θ is the angle formed by the graph of the linear equation and an intersecting horizontal.	$T_{n,c}$: The angle of inclination determines the ratio $(y_2-y_1)/(x_2-x_1)$, which is equivalent to $tan\ \theta$.
Determining Property (D)	Property that determines if lines are parallel or perpendicular; property can determine a line if a point on the line is also given.	$D_{v,p}$: Parallel (perpendicular), coplanar lines have the same slope (negative reciprocal); the slope and a point determine a unique line.	$D_{v,c}$: Parallel lines have the same vertical change for a set horizontal change (otherwise they would intersect); may be seen in terms of congruent slope triangles.
		$D_{n,p}$: Ratio $(y_2-y_1)/(x_2-x_1)$ is equivalent for parallel lines and results in negative reciprocals for perpendicular lines; slope and a point determine a unique linear equation.	$D_{n,c}$: Parallel lines have equivalent differences in *y* values for a set difference in *x* values, yielding equivalent slope ratios.
Calculus Conception (C)	Limit; derivative; a measure of the instantaneous rate of change for any (even nonlinear) functions; tangent line to a curve at a point.	$C_{v,p}$: Slope of a curve at a point is the slope of the tangent line to the curve at a given point.	$C_{v,c}$: Visual interpretation of secant lines approaching tangent line to understand slope of a curve at a point as the slope of the tangent line at the point.
		$C_{n,p}$: Derivative f' is used to calculate slope of the function *f* at a particular point.	$C_{n,c}$: interpreting $f'(x)=\lim_{\Delta x\to 0}(f(x+\Delta x)-f(x))/\Delta x$ as the average rate of change over increasingly small intervals, until arriving at the instantaneous rate of change.

METHOD

This research follows a qualitative and exploratory approach, aiming to identify the components of slope that mathematics teachers employ when tackling a set of six tasks involving this mathematical concept. Data collection was carried out through a Task-Based Interview, following Goldin's framework (2000), albeit adapted to a virtual format due to the exigencies imposed by the health emergency caused by the SARS-CoV-2 virus (COVID-19). This choice is deemed appropriate as it facilitates the identification of mathematical concepts and procedures employed by teachers in task resolution. The Task-Based Interview employed a semi-structured design, necessitating limited interaction between the teacher (task solver) and the interviewer (researcher). During or immediately after the completion of each task (question, problem, or activity), the teacher articulated their thought process, thereby exposing their knowledge and reasoning in approaching the task (Koichu & Harel, 2007)

Participants

The study involved seven voluntary mathematics teachers with high school work experience, hailing from Mexico and Colombia. All participants possessed both experience in teaching the slope concept and a shared interest in pursuing postgraduate studies at the Faculty of Mathematics in Guerrero, Mexico. As shown in Table 2, the participants included four female and three male teachers, among whom four held bachelor's degrees while three held master's degrees. To uphold anonymity and confidentiality, pseudonyms were assigned to each participant.

Table 2
Description of participants

Teacher (age)	Degree of studies	Years of Experience	Country
Raf (25)	Bachelor’s degree in teaching mathematics	1 year	Colombia
Vir (30)	Master of Education	2 years	Mexico
Braw (32)	Bachelor of Mathematics	3 years	Mexico
Dor (31)	Master in educational mathematics	5 years	Mexico
Nic (35)	Bachelor’s degree in mathematics teaching	2 years	Colombia
Mar (56)	Master in mathematics	25 years	Mexico
Irla (32)	Bachelor of Mathematics	2 years	Mexico

Tasks and data collection

The protocol used in the interviews consisted of six tasks. The first two involve the concept of slope in real-world situations in its two approaches—physical situation (task 1) and functional situation (task 2) as defined by Moore-Russo et al. (2011). The primary aim was to ascertain how teachers establish a connection between these scenarios and the concept of slope. The first task was designed by the researchers, wherein an image of a house was presented, and participants were queried regarding the interrelation of the indicated roofs. The second was an adaptation of a situation proposed in the Middle school Mathematics 3 book (Valiente & Valiente, 2004). In this task, participants were tasked with selecting the most suitable computer rental option from three provided quotes. The remaining tasks (Task 3 to 6) were aimed at eliciting insights into what participants deem relevant for the teaching and learning of slope. To achieve this, explicit questions referencing the concept were posed: 3) What thoughts comes to mind upon hearing the term "slope"? 4) What is the slope? 5) What does teaching the slope concept entail? And 6) What is learning the concept of slope?

Each interview was conducted by a researcher, with an average duration of 45 minutes. These interviews were recorded on video (with prior authorization from the participants) using Google Meet. Additionally, the written evidence of the participants' task resolutions was obtained through the WhatsApp application for subsequent analysis. During this process, the interviewer presented the task to the teacher and requested commentary on the ideas and procedures employed during the resolution. Within this interaction, if the interviewer identified a key phrase or procedure relevant to the study's objectives, auxiliary questions were posed. These included inquiries such as: why did you do it this way? Do you know another way to solve it? Could you clarify the meaning of this term? Can you provide examples of how you present this concept in your class? How do you know that they have learned it? among others, with the aim of delving into the justifications and reasoning used by the participants.

Data analysis

For the data analysis, the written productions sent by the teachers were collected and the recorded interviews were transcribed using a word processor (Microsoft Word). To structure the data, a Microsoft Excel spreadsheet was utilized. In the initial column, the teachers' pseudonyms were listed, and the subsequent columns contained the respective answers for each task. This compilation was shared with each researcher for their individual analysis.

The analysis used was a code-by-code approach, entailing the identification of phrases or procedures referencing the previously delineated slope components and subcomponents as described in Table 1. Each researcher autonomously assigned codes to the responses provided by the teachers across the six tasks. Subsequently, they convened to cross-reference the initial codes. In instances of disparities, discussions were held to harmonize the coding approach until a unanimous agreement was attained. In cases where the phrases or procedures deviated from the characterization of any components or subcomponents but still encompassed alternative notions related to the slope concept, a category labeled as "Free (O)" was introduced.

Finally, the researchers examined patterns within the identified codes to formulate the final codes, as described in Table 3. This allowed the reliability of the results and guaranteed their validity, credibility, and rigor, eliminating the bias of a single investigator.

Table 3
Description of the codes corresponding to each slope component

Component	Description of Codes (Teacher Statements)
Constant Ratio (R)	R_1: The slope is the constant rate of change cost per hour in the linear equations. R_2: The slope of the roof or a line is calculated as the vertical displacement between the horizontal displacement in the drawing. R_3: The slope is calculated by applying the formula $(y_2-y_1)/(x_2-x_1)$. R_4: The slope is the variation of y with respect to x in the Cartesian plane.
Trigonometric Conception (T)	T_1: The slope is equal to or calculated with tan θ, where θ is the angle of inclination. T_2: The slope measures the degree of inclination of the roof or objects (straight line). T_3: The slope is the tangent of the angle of inclination which is equivalent to $(y_2-y_1)/(x_2-x_1)$. T_4: The slope is related to the angle of inclination that is formed with respect to the horizontal.
Behavior Indicator (B)	B_1: The slope indicates that if it is positive then the line is increasing and if it is negative it is decreasing.
Determining Property (D)	D_1: The lines are parallel because their slopes are equal.
Free: Unforeseen component (O)	O_1: The slopes are positive or negative. O_2: The slope of a line does not depend on its length. O_3: The slope is m in the linear equations of the form $y = mx+b$ O_4: The slope is positive if it has an angle less than 90º. O_5: The slope is used for decision making. O_6: Slope is associated with real-life situations.

FINDINGS

The findings highlight the components and subcomponents of slope referenced by each of the high school teachers in solving the tasks (see Figure 1). Figure 1 illustrates that the mathematics teachers' understanding regarding the concept of slope tends to be more procedural, with visual interpretations predominating, followed by non-visual ones. Only two teachers were able to manifest a more conceptual understanding in at least one of the components.

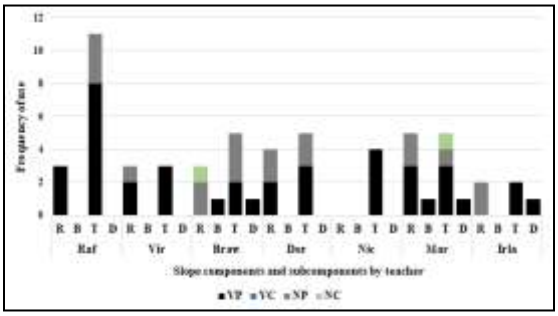

Figure 1
Components and subcomponents of slope utilized by each teacher

Regarding the prevalent components and subcomponents of the slope concept within the knowledge of the seven mathematics teachers, the Trigonometric Conception and Constant Ratio components stood out, emphasizing a procedural visual approach followed by a procedural non-visual one (See Figure 2). The components of Determinant Property and Behavior Indicator appeared with less frequency. In the case of the Calculus Conception component, it did not feature in the arguments and productions of the teachers, despite the utilization of auxiliary questions (for instance, Can you calculate the slope of a curve at a point [by showing them the graph of a non-linear function]?) which provided an opportunity for its expression as well as for delving deeper into the teachers' knowledge.

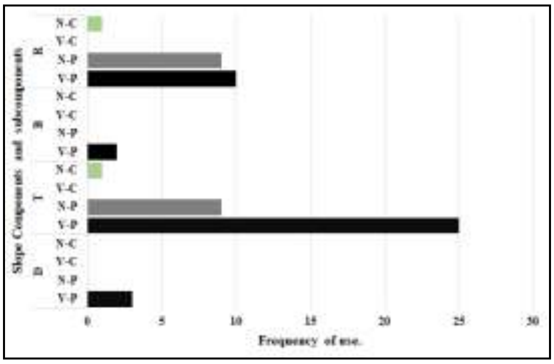

Figure 2
Frequency of slope components and subcomponents

Below, we present the codes associated with the arguments provided by each of the teachers in the tasks, which allowed for the identification of slope components and subcomponents, along with their respective frequency of use (see Table 4). For instance, teacher Rafael referenced code R_4, which pertains to "The slope is the variation of *y* with respect to *x* on the Cartesian plane" (see Table 3). This corresponds to the Constant Ratio component within its visual procedural subcomponent (R_{V-P}), as indicated in the descriptions shown in Table 1.

Table 4
Components and subcomponents of teachers' slope

Teacher	Components and subcomponents													
	R				B		T				D		O	
	v	n			v		v	n			v			
	p	p	c	F	p	F	p	p	c	F	p	F		F
Raf	R_4			3				T_1		3			O_1	1
							T_2			5			O_2	1
							T_4			3			O_6	3
Vir	R_2			2			T_4			3			O_1	2
		R_3		1									O_2	1
													O_4	1
													O_6	4
Braw			R_1	1	B_1	1	T_4			2	D_1	1	O_2	1
		R_3		2				T_1		3			O_3	1
													O_6	4
Dor		R_3		2			T_2			1			O_1	3
	R_2			1			T_4			2			O_3	1
	R_4			1				T_1		2			O_5	1
													O_6	1
Nic							T_4			4			O_2	1
													O_3	1
													O_5	1
													O_6	3
Mar	R_2			2	B_1	1	T_2			1	D_1	1	O_2	1
	R_4			1			T_4			2			O_4	1
		R_3		2				T_1		1			O_6	3
									T_3	1				
Irla		R_3		2			T_4			2	D_1	1	O_1	3
													O_2	1
													O_6	2

Note: v refers to visual, n to non-visual, p to procedural, c to conceptual, and F to frequency of use.

The following results are organized thematically, with interview excerpts to illustrate the components that constitute the mathematics teachers' knowledge.

Constant Ratio component (R)

This slope component was identified in the statements of six teachers. Four included procedural visual interpretations (R_4, R_2), three teachers included the non-visual procedural subcomponent (R_3), and only one teacher included the non-visual conceptual subcomponent (R_1). Although all responses referred to slope as a Constant Ratio, the non-visual procedural subcomponent, which involves using the formula $(y_2-y_1)/(x_2-x_1)$ stood out more prominently. On the other hand, it was observed that three teachers approach slope both visually and non-visual, but from a procedural standpoint. This provides evidence of a connection between the algebraic formula and a representation on the Cartesian plane, as evident in an excerpt from teacher Mari's response in Task 5:

Interviewer: What is teaching the concept of slope?

Mar: To teach the concept of slope is to demonstrate that when advancing a certain measure vertically there was an advance horizontally and then when we continue this process and connect the starting point with the endpoint of this movement (demonstrated by simulating elevation and sequential advancement with hands by tracing a segment of straight into the air), we can observe the emergence of a straight line with a specific degree of inclination. This can be verified with the formula *y* two minus *y* one divided by *x* two minus *x* one. This entire concept becomes apparent in the construction of stairs, where both vertical and horizontal displacements are present.

Trigonometric Conception component (T)

For this component, the statements of the seven teachers refer to the slope being related to the angle of inclination formed with respect to the horizontal, which emphasizes a more visual procedural approach (T_2, T_4). Of these seven, three of them added that the slope measures the degree of incline of a line and/or physical situations, such as: on the roofs of houses (Task 2), hills, mountains, pendulums on a bridge, stairs, or steep streets. Furthermore, four teachers demonstrated the non-visual procedural subcomponent of slope (T_1), as they defined, calculated, and referred to the slope as the tangent of the angle of inclination. Only one teacher associated tangent of the inclination angle with the formula $(y_2-y_1)/(x_2-x_1)$, showing a conceptual non-visual approach.

In general, for this component, four teachers connected both the visual and non-visual approaches, with an emphasis on the procedural aspect, as shown in the following excerpt:

Interviewer: What is slope?

Raf: Slope is the degree of incline that a straight line presents in relation to a fixed axis (positioning the arm horizontally). This incline is calculated using m=tan θ, where θ is the angle of inclination of the line.

On the other hand, only teacher Mar possesses this component with three subcomponents, as she was the sole individual to emphasize a non-visual conceptual approach, which refers to the connection between m=tan θ and the formula $(y_2-y_1)/(x_2-x_1)$. In her discourse, she mentions that: "The tangent of θ is defined as the opposite side divided by the adjacent side, which is the same, because the measure of the opposite side is y_2-y_1 and the measure of the adjacent side is x_2-x_1".

Determinant Property component (D)

This component was only referenced in its visual procedural subcomponent (D_1) by three teachers. It occurred when determining the parallelism between two roofs based on the values of their respective slopes, indicating their equality. It's noteworthy that even though the term "slope" was not explicitly mentioned in the task, the teachers extrapolated the situation to a mathematical context by associating the roofs with straight lines and thereby connecting them with the slope concept. This can be observed in the interview excerpt from teacher Braw.

Interviewer: On the facade of the house, there are roofs A, B, and C. How are they related to each other?

Braw: If we see the roofs as multiple straight lines, they are parallel.

Interviewer: Why?

Braw: Yes, because if we put a Cartesian plane, then the roofs would be straight lines, and it can be observed that they have the same slopes.

Interviewer: How do you know they have the same slope?

Braw: Because if we consider the angle of inclination formed with the x-axis, it's the same. So, their slopes are equal.

Behavior Indicator component (B)

This component was highlighted in a visual procedural approach (B_1) by two out of the seven teachers, associating an increasing (or decreasing) line with the positive (or negative) slope. It was primarily identified in Task 1 when comparing the roofs of a house with a straight line, as shown in the excerpt from Mar's interview.

Interviewer: On the facade of the house, there are roofs A, B and C. How are they related to each other?

Mar: If we see the roofs as straight lines on the Cartesian plane, B and C have a negative slope, and roof A has a positive slope. I also observe that the slopes are nearly the same (pointing to roof B and C), and the slope of roof A is smaller... So, it´s clear that two of them have negative slopes, indicating decreasing lines, while one has a positive slope, indicating an increasing line.

It is worth mentioning that the procedural aspect becomes evident due to the need to imaginary straight lines over the roofs to work with them in a mathematical context and thus declare the behavior of said lines through their slopes. This allowed for inferences about the limited treatment of slope in real-life situations, which leads teacher Mar and Braw to overlook the original context of the task.

No component (free)(O)

This category refers to the set of ideas associated with the concept of slope that do not meet the descriptions of the components and their subcomponents, but there are indications of a knowledge about the concept. For instance, four teachers are aware that the slope can be positive or negative (O_1), which showed a visual approach due to the context presented in Task 1. Similarly, only two teachers consider that a slope will be positive if it forms an angle of inclination less than 90º with respect to the horizontal (O_4).

On the other hand, six teachers have recognized that extending a straight line does not affect the value of its slope (O_2), since it is a constant that guarantees its linearity, as can be seen in the teacher Braw's excerpt.

Interviewer: If the length of roof A increases, what will happen to its slope?

Braw: The slope will remain the same, the slope is preserved, as I said, if we see the roof as a straight line, extending it would mean we're elongating the line and we know its properties are preserved.

Interviewer: Which properties are you referring to?

Braw: Its inclination, meaning that it continues to be maintained, and therefore its slope does as well.

Another idea allowed us to identify that in three teachers, a non-visual approach to slope predominates, emphasizing that *m* is in the linear equations whose algebraic expression is $y = mx + b$ (O_3), without delving into the connection between the sign of the *m* value and its importance in the behavior of functions.

Interviewer: ... Which company is it better to do business with?

Dor: The one with the lower slope.

Interviewer: Why?

Dor: because if we represent each budget with linear equations, in this case, *y* equal to *mx* plus *b,* we know that *m* is the slope. So here, the equation would be *y* equal to one thousand five hundred plus twenty *x* (write on the worksheet all the equations and make their respective graphs) and so on with the others... and here, if we see that the slope is lower compared to those slopes of the other equations, the result is going to be lower.

The idea of slope as a tool for decision-making (O_5) was mainly featured in Task 2 by two teachers. Teacher Nic solely approached the task using a non-visual approach, relying on the algebraic expressions that model the given budgets. He indicated that by comparing the slope values, he could decide. Similarly, teacher Dori suggested this, but she combined it with a visual approach. She proposed creating the corresponding graphs to visualize how the lines and their slopes behave. Finally, the slope associated with real-life situations (O_6) is an idea that all the teachers expressed, primarily in Tasks 5 and 6, which refer to what it means to teach and learn the concept of slope. Within their arguments, it was identified that the most common situations they associate with slope are those referred to as physical situations, that is, roofs, streets, roads, stairs, mountains, and/or pendulum of a bridge.

DISCUSSION

The analysis of the results highlights that the components and subcomponents of slope, constitute a part of the knowledge held by this group of teachers, stem from the analytical definition of the concept regularly found in textbooks. This outcome aligns with the findings reported by Salgado et al. (2019), as they deduced from the analysis of student class notes that, in high school, teachers teach, explain, and propose activities related to the use of *tan* θ, as well as the formula $(y_2-y_1)/(x_2-x_1)$. These elements correspond to the Trigonometric Conception and Constant Ratio components in terms of their components. This result slightly differs from what is observed in high school students, as reported by Rivera et al. (2019). These students show limited recognition of the slope as *tan* θ (trigonometric component-non-visual procedural subcomponent). Instead, there is a prevalent understanding of slope as the incline of a line (trigonometric component-visual procedural subcomponent), as well as the algebraic ratio $(y_2-y_1)/(x_2-x_1)$ (Constant Ratio component-non-visual procedural subcomponent).

The visual conceptual subcomponent of all components did not emerge in the teachers' responses, whereas the non-visual conceptual aspect was sporadically referenced by two teachers (Figure 2). This underscores the notion that the treatment of the concept leans heavily towards the procedural, sidelining the conceptual aspect, as also identified by Lingefjärd and Farahani (2017), Styers et al. (2020) and, Salgado and Dolores (2021). Such an approach poses a hindrance to establishing connections among rate of change, slope, and steepness, a concern highlighted by prior studies (Dolores et al., 2019; Nagle & Moore-Russo, 2013; Teuscher & Reys, 2012).

Another important aspect to note is that when referring to the concept of slope, the teachers emphasize a connection with real-life or everyday situations. For instance, they associate it with ramps, stairs, steep streets, roofs of houses, among others. Similar findings have been reported by Hoffman (2015) and Avcu and Biber (2022) with high school teachers and prospective teachers, respectively. However, this study inferred that the emphasis they place on these connections might often be a memorized discourse. When confronted with activities involving real-life contexts, teachers tend to decontextualize the situation and work with it purely in a mathematical context. It was observed that they do not internalize or reflect upon the meaning of the concept in those situations. This leads to the hypothesis that the attempt to highlight the application of mathematics could be influenced by the curriculum. According to Dolores et al. (2020) and Dolores and Mosquera (2022), curriculum guidelines suggest associating mathematical concepts with everyday life.

In the Mexican curriculum, there isn't a precise specification regarding the type of situations to use, whether they are physical or functional in nature. In contrast, the Colombian curriculum emphasizes variational ideas, which could be associated with functional situations. However, the results from this study indicate that Colombian teachers place little emphasis on these ideas. For instance, in Task 2, which involved a functional situation, five of the teachers translated the common language into algebraic expressions ($y=mx+b$) to derive linear equations that represented the given conditions in the situation. Consequently, they performed only arithmetic calculations to compare *y* values for the same *x* value, omitting the recognition and interpretation of the slope within the scenario. This omission prevented them from considering the slope when making decisions.

Another teacher linked the situation with the slope by the fact that the algebraic expressions represent straight lines in the Cartesian plane and therefore one can talk about their slopes and compare their steepness. This finding is similar to what Rivera and Dolores (2021) discovered in high school students, who believe that a situation or problem is related to the slope only if the graphed data forms a straight line. This highlights that the slope as an inclination has a greater influence on the usage and meaning attributed by both teachers and students. This aspect, as per Diamond (2020), could be another factor that constrains the comprehension of the connection between slope and the rate of change, in other words, interpreting it in its variational notion.

The definition of slope provided by teachers, as well as the one given by high school and middle school students, are consistent and fall within the Trigonometric Conception component, specifically the visual-procedural subcomponent. This component refers to slope as the inclination of a line relative to the horizontal axis (Rivera et al., 2019; Rivera & Dolores, 2021; Salgado et al., 2019; Salgado & Dolores, 2021). In this regard, Avcu and Biber (2022) report that prospective high school teachers face challenges in defining slope mathematically but can do so colloquially, describing it as the angle of inclination. However, this shouldn't be regarded as an error, as this idea contributes to the understanding of slope as the tangent of the angle of inclination, provided this transition is made correctly.

As can be observed, the results continue to suggest the need to question the quality of training and updates that teachers are receiving, in order to propose strategies for achieving a stronger foundation of their knowledge. Nowadays, when updating curriculum plans and programs, little consideration is given to whether teachers' knowledge is sufficient to address these curricular demands (Luján, 2021). In the case of Mathematics Education, the aim is always to ensure that teaching and learning are relevant and encompass the challenges and opportunities of real life (Suntusia et al., 2019).

CONCLUSION

The present study aimed to address the research question: What components and subcomponents of slope do high school teachers emphasize when solving tasks involving slope? It was found that teachers emphasized the components of Constant Ratio and Trigonometric Conception, both in their visual-procedural and non-visual-procedural subcomponents. This suggests that in the teachers' knowledge, there is a neglect of variational ideas concerning the concept of slope. Their responses did not provide evidence of reflection on fundamental questions when working with tasks involving variation, specifically, questions such as What is changing? and What is it changing with respect to? The absence of these ideas, as well as the focus on conceptual ideas, makes it challenging for them to establish connections between subcomponents and even more so between components.

Regarding the curriculum demands, particularly when it comes to applying or solving real-life situations involving the concept, it is concluded that the teachers' knowledge hinders their ability to recognize it in real-life situations where the term "slope" is not explicitly mentioned, especially in cases involving functional situations. Teachers are more familiar with physical situations. However, their interpretations in this context are often incorrect, which may be a consequence of their procedural understanding of the concept.

In general, this study once again confirms the teachers' weak understanding of the concept of slope, as they struggle to establish connections between the various components and subcomponents of the concept. Concerning their teaching practices, a correspondence was identified between what they claim to teach and what they consider necessary for assessing their students' learning. Their focus tends to be on algorithmic practices, with little emphasis on the analysis of its real-world applications and interpretations in everyday life.

LIMITATIONS

This study included a relatively small number of participants for two reasons. Firstly, the study was conducted during the COVID-19 pandemic, which limited the availability of teachers for research participation. Secondly, teachers were in the process of adapting to the new mode of teaching, which made them less receptive to participating in Task-Based Interview. However, it's important to note that the research context is not believed to have significantly influenced the results. During the interviews, teachers were asked about the challenges they faced when teaching the concept in a virtual modality. At least six of the teachers mentioned that there wasn't much difference, as they perceived that the only change was the mode of teaching content. From their perspective, they tried to cover the content as if it were in a face-to-face setting, and the difficulties they encountered were primarily technical in nature.

Furthermore, the study's results highlight the need to support teachers in expanding their understanding and interpretation of slope in authentic and realistic scenarios, encompassing both geometric and variational notions. Hence, in terms of future research directions, there is potential for focusing on the analysis of a teacher's class during the teaching of the concept. This analysis can help identify ways to develop a proposal that fosters connections among the components and subcomponents of the slope concept, drawing inspiration from the ideas explored by teachers. Additionally, within the line of teacher professional development research, it is feasible to create a course tailored for secondary and high school teachers, focusing on the exploration of connections among the various components. Similarly, a workshop that supports teachers in designing tasks that facilitate and scaffold student understanding, enabling them to achieve meaningful learning.

REFERENCES

Avcu, S., & Biber, B. (2022). Prospective middle school mathematics teachers' conceptualizations of slope. *International Journal of Mathematical Education in*

Science and Technology, *53*(11), 3115-3137. https://doi.org/10.1080/0020739X.2022.2073478

Cho, P., & Nagle, C. (2017). Procedural and Conceptual Difficulties with Slope: An Analysis of Students' Mistakes on Routine Tasks. *International Journal of Research in Education and Science*, *3*(1), 135-150.

Diamond, J. M. (2020). Mathematical Knowledge for Teaching Slope: Leveraging an Intrinsic Approach. *Investigations in Mathematics Learning*, *12*(3), 163-178. https://doi.org/10.1080/19477503.2020.1754091

Dolores, C., Rivera, M. I., & García, J. (2019). Exploring mathematical connections of pre-university students through tasks involving rates of change. *International Journal of Mathematical Education in Science and Technology*, *50*(3), 369-389. https://doi.org/10.1080/0020739X.2018.1507050

Dolores, C., Rivera, M. I., & Moore-Russo, D. (2020). Conceptualizations of slope in Mexican intended curriculum. *School Science and Mathematics*, *120*(2), 104-115. https://doi.org/10.1111/ssm.12389

Dolores, C., & Mosquera, G. A. (2022). Conceptualizaciones de la pendiente en el currículum colombiano de matemáticas. *Educación Matemática*, *34*(2), 217-244. https://doi.org/10.24844/em3402.08

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. Kelly & R. Lesh. (eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 517-545).

Hoban, R. A. (2021). Introducing the slope concept. *International Journal of Mathematical Education in Science and Technology*, *52*(6), 879-895. https://doi.org/10.1080/0020739X.2020.1723726

Ismail, S. A. A., & Jarrah, A. M. (2019). Exploring Pre-Service Teachers' Perceptions of Their Pedagogical Preferences, Teaching Competence and Motivation. *International Journal of Instruction*, *12*(1), 493-510. https://doi.org/10.29333/iji.2019.12132a

Koichu, B., & Harel, G. (2007). Triadic interaction in clinical task-based interviews with mathematics teachers. *Educational Studies in Mathematics*, *65*(1), 349-365. https://doi.org/10.1007/s10649-006-9054-0

Lingefjärd, T., & Farahani, D. (2017). The symbolic world of mathematics. *Journal of Research in Mathematics Education*, *6*(2), 118-135. https://doi.org/10.17583/redimat.2017.2391

Lobato, J., & Thanheiser, E. (2002). Developing understanding of ratio-as-measure as a foundation for slope. In B. Litwiller & G. Bright. (eds.). *Making sense of fractions, ratios, and proportions* (pp. 162- 175).

Luján, E. L. (2021). The beliefs of primary school teachers: A comparative analysis. *International Journal of Instruction*, *14*(3), 223-240. https://doi.org/10.29333/iji.2021.14313a

Moore-Russo, D., Conner, A., & Rugg, K. I. (2011). Can slope be negative in 3-space? Studying concept image of slope through collective definition construction. *Educational studies in Mathematics*, *76*(1), 3-21. https://doi.org/10.1007/s10649-010-9277-y

Nagle, C., & Moore-Russo, D. (2013a). The concept of slope: Comparing teachers' concept images and instructional content. *Investigations in Mathematics Learning*, *6*(2), 1-18. https://doi.org/10.1080/24727466.2013.11790330

Nagle, C. & Moore-Russo, D. (2013b). Slope: A network of connected components. In M. Martínez & A. (eds.). *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 127-135).

Nagle, C., Moore-Russo, D., Viglietti, J., & Martin, K. (2013). Calculus students' and instructors' conceptualizations of slope: A comparison across academic levels. *International Journal of Science and Mathematics Education*, *11*(1), 1491-1515. https://doi.org/10.1007/s10763-013-9411-2

Nagle, C., Moore-Russo, D., & Fisher, T. (2022). A Study of Slope Explanations across a Standards-Based Textbook Series. *International Journal for Mathematics Teaching and Learning*, *23*(1), 25-45. https://doi.org/10.4256/ijmtl.v23i1.403

Paolucci, C., & Stepp, Z. A. (2021). Examining preservice teachers' understanding of slope through posing problems and embedding learning in real-world contexts. *Teaching and Teacher Education*, *107*(1), 103476. https://doi.org/10.1016/j.tate.2021.103476

Rivera, M. I., & Dolores, C. (2021). Preconcepciones de pendiente en estudiantes de Educación Secundaria. *Enseñanza de las Ciencias*, *39*(1), 195-217. https://doi.org/10.5565/rev/ensciencias.3045

Rivera, M. I., Salgado, G. & Dolores, C. (2019). Explorando las conceptualizaciones de la pendiente en estudiantes universitarios. *Bolema*, *33(65)*, 1027-1046. https://doi.org/10.1590/1980-4415v33n65a03

Salgado, G., Rivera, M. I., & Dolores, C. (2019). Conceptualizaciones de pendiente: Contenido que enseñan los profesores del Bachillerato. *UNIÓN-Revista Iberoamericana de Educación Matemática*, *15*(57), 41-56.

Salgado, G., & Dolores, C. (2021). Imagen del concepto de pendiente evocado por profesores del bachillerato. *Números*, *109*(1), 89-109.

Sharp, L. A., Bonjour, G. L., & Cox, E. (2019). Implementing the Math Workshop Approach: An Examination of Perspectives among Elementary, Middle, and High School Teachers. *International Journal of Instruction*, *12*(1), 69-82. https://doi.org/10.29333/iji.2019.1215a

Stanton, M., & Moore-Russo, D. (2012). Conceptualizations of slope: A review of state standards. *School Science and Mathematics*, *112*(5), 270-277. https://doi.org/10.1111/j.1949-8594.2012.00135.x

Stump, S. (1999). Secondary mathematics teachers' knowledge of slope. *Mathematics Education Research Journal*, *11*(2), 124-144. https://doi.org/10.1007/BF03217065

Stump, S. L. (2001). Developing preservice teachers' pedagogical content knowledge of slope. *The Journal of Mathematical Behavior*, *20*(2), 207-227. https://doi.org/10.1016/S0732-3123(01)00071-2

Styers, J., Nagle, C., & Moore-Russo, D. (2020). Teachers' noticing of students' slope statements: attending and interpreting. *Canadian Journal of Science, Mathematics and Technology Education*, *20*(1), 504-520. https://doi.org/10.1007/s42330-020-00107-5

Suntusia, Dafik, & Hobri. (2019). The Effectiveness of Research Based Learning in Improving Students' Achievement in Solving Two-Dimensional Arithmetic Sequence Problems. *International Journal of Instruction*, *12*(1), 17-32. https://doi.org/10.29333/iji.2019.1212a

Teuscher, D., & Reys, R. E. (2012). Rate of change: AP calculus students' understandings and misconceptions after completing different curricular paths. *School Science and Mathematics*, *112*(6), 359-376. https://doi.org/10.1111/j.1949-8594.2012.00150.x

Teuscher, D., & Reys, R. E. (2010). Connecting research to teaching: Slope, rate of change, and steepness: Do students understand these concepts? *The Mathematics Teacher*, *103*(7), 519-524. https://doi.org/10.5951/MT.103.7.0519

Valiente, S., & Valiente, S. (2004). *Matemáticas 3: Hechos, conceptos y problemas en el aprendizaje de las Matemáticas*. Ediciones Castillo.