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Characterising Pre-Service Primary School Teachers' Discursive Activity when Defining

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This paper studies how pre-service primary school teachers construct and select mathematical definitions through the analysis of their discursive activity. Specifically, the theory of commognition (Sfard, 2008) is employed to determine whether the existence of different meta-rules always leads to the existence of a commognitive conflict. Moreover, we study the reasons that give rise to the commognitive conflicts found and whether they are resolved. To this end, we studied the discourse of 45 pre-service primary school teachers while they answered several questions on defining geometric solids. The data in this study consisted of audio recordings of their discussions and their written answers. In this paper, three vignettes showing different meta-rules are presented. In the first, discussions regarding the characteristics of a definition promoted the appearance of different meta-rules that existed in incommensurable discourses, which meant the existence of a commognitive conflict. This conflict highlights the fact that certain pre-service teachers confuse the processes of describing and defining. Both the second and third vignettes featured the appearance of two different meta-rules. However, in both cases, those meta-rules could coexist in the same discourse, and therefore a commognitive conflict could not be inferred.

Keywords: commognitive framework, commognitive conflict, discourse, mathematical practice of defining, meta-rule, pre-service primary school teacher

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INTRODUCTION

The practice of defining is essential in mathematics. It is as important as finding the proof of a theorem (Freudenthal, 1973), and learning the relevant definitions should hence form an essential part of the process of learning mathematics (Avcu, 2022; Miller, 2018; Tabach & Nachlieli, 2015; Zaslavsky & Shir, 2005). Therefore, learning to define should constitute an important part of mathematics education (Mariotti & Fishbein, 1997; Zaslavsky & Shir, 2005), but this mathematical practice is often neglected (de Villiers, 1998) and remains "typically underemphasized in school mathematics; instead, students often experience definitions as received from an authority" (Kobiela & Lehrer, 2015, p. 425).

Learning to define should also form an important part of mathematics teacher education (Leikin & Zazkis, 2010; Miller, 2018; Sánchez & García, 2008) and, during that learning, "explicit connections [should] be made between university mathematics and school mathematics" (Leikin & Zazkis, 2010, p. 465). Indeed, it is very important that pre-service teachers learn how to use or propose definitions because that knowledge may influence their future choice of pedagogical approaches to teaching mathematical definitions (Avcu, 2022). In geometry, a field in which mathematical communication skills are particularly needed (Kusumah et al., 2020), teachers should encourage students to participate in the process of defining because "mathematicians and mathematics educators alike have often criticised the direct teaching of geometry definitions with no emphasis on the underlying process of defining" (de Villiers et al., 2009, p. 190). In order to be able to evaluate the definitions that students propose, teachers should be able to grasp the differences between different types of definitions (de Villiers et al., 2009).

Unfortunately, previous studies report that mathematical definitions constitute a difficult topic for pre-service teachers (e.g., Miller, 2018), so it is important to study the definitions they produce to gain more information regarding their knowledge of mathematical definitions (Avcu, 2022). To this end, Miller (2018) categorised the content and the form of pre-service primary school teachers' (PPTs') definitions of quadrilaterals and Avcu (2022) categorised middle school mathematics teachers' definitions of certain quadrilaterals by using Zazkis and Leikin's (2008) framework. The definitions of three-dimensional objects have been much less explored than the definitions of two-dimensional objects, thus one of the objectives of our study was to fill this gap by studying the definitions that PPTs propose for three-dimensional geometrical objects. According to Tall (1991), one of the aspects that characterises the transition from elementary to advanced mathematical thinking is the transition from describing to defining, and hence the objective included determining whether PPTs distinguished between describing and defining the three-dimensional geometrical solids when proposing their definitions.

One way to obtain information on someone's conception of a mathematical definition is to present them with several definitions of mathematical objects and ask them to choose which one they would prefer as a definition (Zaslavsky & Shir, 2005). Moreover, the ability to access information in our current society means that in-service and pre-service teachers have access to a wide variety of definitions (some of them incorrect) and they have to decide which one is the most suitable for their classroom. These two reasons motivated another objective of our study, which involves determining which criteria PPTs use when they have to choose a definition.

In summary, our study focuses on characterising how pre-service primary school teachers defined and selected definitions for three-dimensional mathematical objects.

Context And Theoretical Background

The mathematical practice of defining, as other mathematical practices, is a social practice which is essential in mathematical knowledge construction (Rasmussen et al., 2005). This social nature makes sociocultural approaches conducive to investigating this mathematical practice. Among the various sociocultural approaches, the theory of commognition (Sfard, 2008) has proven useful to study this and other mathematical practices. For instance, this theory has been used in the study of mathematical practices such as modelling (Viirman & Nardi, 2019) and proving (Schüler-Meyer, 2020). In the case of the practice of defining, this sociocultural approach has recently been employed in the studies by Biza (2021), Fernández-León et al. (2021), Schüler-Meyer (2020), and Tabach and Nachlieli (2015). It should be mentioned that this theoretical perspective has led to many other major results in mathematics education (see, for instance, Gallego-Sánchez et al., 2022; Ioannou, 2018; Nachlieli and Tabach, 2022; Nardi et al., 2014; Sfard, 2021b; Thoma & Nardi, 2017).

In the following, the theory of commognition is described as an operational framework for the study of students' discursive activity when defining.

Sfard (2008) coined the term "commognition" by joining the terms "communication" and "cognition". The theory of commognition regards thinking as a particular type of interpersonal communication; specifically, the communication that one has with oneself. Furthermore, this theory considers mathematics as a particular type of discourse that can be characterised through four characteristics: keywords, visual mediators, narratives, and routines.

The first characteristic, that of keywords, refers both to mathematical words (such as angle and prism) and to colloquial words with a mathematical meaning (such as "leaning" to mean oblique). The visual mediators of physical entities are physical objects (such as a graph or an equation) that participants in the discourse (discursants) use as part of their communication to clarify their performances. Another important feature that defines the mathematical discourse are its narratives, which are sentences (spoken or written) about objects, relations between them, or activities with or by objects. Unlike other discourses, the objects in mathematics, called mathematical objects, do not exist independently of the discourse, but such objects are brought into being when the discursants talk about them (Sfard, 2021b). Furthermore, if a narrative is accepted by the discursants, that is, it is regarded as true, then the narrative is called an "endorsed narrative" (such as definitions or theorems). The fourth and last characteristic that defines the mathematical discourse are the routines, which are repetitive patterns that characterise the actions of the participants in the discourse (such as the way they solve equations or the strategies they employ to prove theorems). In Lavie et al. (2019), routines are described as a task-procedure pair, in which the task captures information about how the task performer views a certain task situation and the procedure refers to the actions this performer undertakes to tackle the task situation. Note that, in this context, a task situation is "any setting in which a person considers herself bound to act – to do something" (Lavie et al., 2019, p. 159). We emphasise here that the task focuses on the performer's interpretation, whereas the procedure focuses on what the researcher interprets from the task performer's actions in a given task situation (Nachlieli & Tabach, 2022).

In the theory of commognition (Sfard, 2008), human communication is defined as an activity governed by rules in which the participants act and react in an organised way according to a well-determined set of options. Sfard (2008) establishes that two types of rules can be identified in human communication: object-level rules and meta-level rules (henceforth, meta-rules). Object-level rules are rules regarding the regular behaviour of objects (such as "the sum of the angles of a square is 360º") and meta-rules are rules that appear when the discursants try to produce or justify narratives at object level (such as "to prove the previous statement, it is appropriate to divide the quadrilateral into two triangles along one of its diagonals").

In this theory, learning is considered a change in the discourse and is manifested by changes in any of the four characteristics of the discourse. These changes may be caused by the resolution of commognitive conflicts, which are situations that arise when the discursants are participants of incommensurable discourses. Incommensurable discourses are those governed by different meta-rules (Sfard, 2021a) and "that differ in their use of words and mediators or in their routines" (Sfard, 2008, p. 299). In other words, commognitive conflicts emerge in those scenarios where participants of incommensurable discourses try to communicate across such discourses, which may produce a feeling of uneasiness (Sfard, 2021b). The study of commognitive conflicts is an outstanding topic that has increased its presence in recent investigations (González-Regaña et al., 2021; Ioannou, 2018; Nachlieli & Heyd-Metzuyanim, 2022; Sánchez & García, 2014) since commognitive conflicts constitute a major source of mathematical learning (Sfard, 2021b).

In this paper, the great potential of certain theoretical constructs of the commognitive framework has been leveraged to characterise PPTs' activity when defining and selecting definitions. To be precise, the research questions that inform about the particular aims of this study are: (1) did the existence of different meta-rules always lead to the existence of a commognitive conflict? (2) When there was a commognitive conflict, what were the reasons that gave rise to it and was it resolved?

METHOD

Participants and context

The participants in this study consisted of 45 PPTs enrolled in an undergraduate degree in Primary Education. Specifically, these pre-service teachers were taking a first-year mathematics course that each week had a two-hour session on theory and exercises and a one-hour session on problem solving. In problem-solving sessions, the students worked in groups of 3 or 4 members.

In one of the problem-solving sessions, the data collection instrument was presented, and the students were asked to collaborate in the study. There were 45 students who voluntarily decided to participate. These students were organised into 12 working groups (called G1, G2, …, G12 in the study), each comprising 3-4 students.

Data collection instrument

The main objective of the data collection instrument, as mentioned in Fernández-León et al. (2021) and González-Regaña et al. (2021), was to promote PPTs' discussion when defining and selecting geometric solids. The data collection instrument was of the form of a worksheet and included a two-dimensional representation of three geometric solids that was designed with the dynamic software GeoGebra (Figure 1). The geometric solids were a cube, a quadrangular, oblique, and convex prism, and a quadrangular, oblique, and concave prism. These were chosen because they have some properties in common (for example, all are quadrangular prisms) and hold other distinctive properties (for example, the cube is regular, while the others are not).

The worksheet also included nine questions. The first four questions were related to the description of geometric solids and the identification of their similarities and differences:

1. In the three previous solids, you can identify basic elements such as faces, vertices, edges, etc. What properties or characteristics of these elements can you observe in each solid?

2. From among the above properties or characteristics, can you identify those that only two solids have in common?

3. Among the properties or characteristics of Question 1, can you identify any property that the three solids have in common?

4. Is there any property of any of the solids that differentiates it from the other two?

The following four questions dealt with the construction of definitions of these solids and reflection thereon.

5. Define each of these solids.

6. Can you give another definition of any of the solids?

7. Is one of your definitions valid for another solid as well? For example, is the definition of solid 1 also valid for solids 2 or 3?

8. Could you give a definition that is valid for two of the solids? And for three?

Finally, there was a question where the PPTs had to select a definition from among those they had previously built.

9. Of the two definitions that you have given for each solid in Questions 5 and 6, which one would you choose? Why?

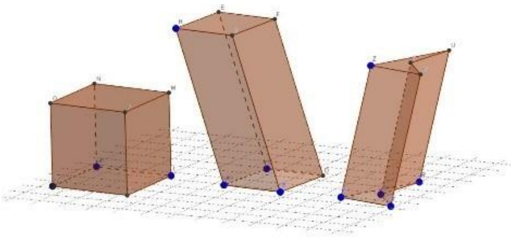

Figure 1
Representation of the three geometric solids included in the worksheet (Fernández-León et al., 2021, p. 306)

Data collection

Data collection took place during a one-hour problem-solving session. A paper copy of the worksheet was given to each group and the PPTs were asked to discuss the proposed questions and to reach an agreement, which they had to write down on the paper copy. Additionally, each of the 12 groups received an audio recorder and instructions to facilitate their identification in the transcript. At the end of the session, 12 paper copies with the written answers of each group were collected, together with the audio recorders with their conversations (approximately 12 hours, 1 hour per group).

Analysis

Once the audio recordings were transcribed, the analysis was carried out in two phases. In the first phase, the transcripts and written answers were analysed to identify categories in keywords, narratives, and visual mediators. These categories were based on similarities and differences found within each characteristic. Such categories were then analysed to identify meta-rules and routines in the discourse. In the second phase, based on the categories obtained in the first phase, the discourse was analysed in search of possible commognitive conflicts. This was carried out by locating differences between the discursive characteristics of the PPTs of each group, and then those differences were carefully analysed to decide whether commognitive conflicts could be inferred. Furthermore, the conflicts that appeared were analysed to determine the reasons that led to them and whether they had been resolved.

Findings obtained in the first phase of analysis can be found in previous work (Fernández-León et al., 2021; Gavilán-Izquierdo et al., 2019; González-Regaña et al., 2021). In this paper, we focus on the findings obtained in the second phase of the analysis.

FINDINGS

In this section, three vignettes are presented, showing situations in which three groups of PPTs employed different meta-rules while they were answering the questions of the worksheet. In all three vignettes, more than one meta-rule appeared, but a commognitive conflict appeared in only one of them and that conflict remained unresolved. All the vignettes are described in detail below, including excerpts from the transcripts and written answers, and the context in which the discussion occurs. In each excerpt, the PPTs are referred to as S1, …, S4, regardless of the group to which they belong, and their teacher is referred to as T.

Vignette 1. Defining versus describing

The task situation created by the fifth question of the worksheet required the PPTs to define the three solids for the first time, while the previous four questions asked them to describe the solids. No information regarding the structure or characteristics of a definition was provided to the PPTs to determine whether they distinguished between defining and describing. One of the discussions concerning how to answer Question 5 of the worksheet appears in the following excerpt from the transcript of group G7:

207 S1: If we talk about prisms, it is a hexagonal prism, but…
208 S4: We have already defined it, in…
209 S3: No, we have said what it has.
210 S4: Exactly, the characteristics. Sure, that it is a cube with these characteristics…
211 S2: Teacher, what do we have to write in Question 5?
212 T: My question to you is: what do you consider a definition to be? Saying which are the characteristics? Saying the name? What?
213 S2: The name and… the characteristics… is the definition.
214 T: That is what I am asking you.
215 S1: I think so…
216 T: Then, talk about it among yourselves and what you decide is what you have to write down.

In this excerpt, four PPTs discussed what information should be included in the definition of the first solid. The misalignment between the different meta-rules that they used to construct such a definition has allowed us to infer the existence of a commognitive conflict between the discourses of the PPTs. On the one hand, line 208 includes a meta-level narrative that shows what S4 understood by process of defining, which he considered equivalent to describing. In the next line, a meta-level narrative from S3 shows a different meta-rule to define that contradicts S4's meta-rule, since S3 stated in line 209 that the description given in Question 1 was not a definition. From this last narrative, a discussion arose and, since the PPTs did not come to an agreement, they finally turned to the teacher (an expert) asking for her help. However, the teacher did not explicitly tell them what they had to write down, but rather motivated them to reflect on the meta-rules that they might consider when defining (lines 212, 214, and 216). Finally, students S1 and S2 adopted a common meta-rule. This meta-rule has been inferred from the narrative at meta level that appears in line 213, in which S2 claimed that a definition of a solid should include its name and its characteristics. A resolution of the commognitive conflict cannot be deduced from the data since the PPTs stopped their discussion and simply wrote down the following answer to Question 5: "Solid 1: A cube is a geometric body with six equal faces with a square base whose angles are right angles".

Vignettes 2 and 3. What is the best definition? PPTs' meta-choices

In the task situation generated by Question 9 of the worksheet, the PPTs had to choose a definition for each solid from the ones they had been asked to construct in Questions 5 and 6. They also had to justify their choices, which generated discussions in several of the groups. The following vignettes show two situations where different meta-rules do not lead to the existence of a commognitive conflict.

In vignette 2, the PPTs in group G11 had the following discussion when justifying their chosen definition:

456 S1: Because it is the most complete one.
457 S3: Sure, here we have completed it more.
458 S4: We have chosen this definition because it is the most complete. You said something before.
459 S1: About what?
460 S4: About this definition. You said that…
461 S1: …that it is the most specific one.
462 S4: Should I write that it is the most complete and specific?

In the previous excerpt, all the PPTs agreed on which definition to choose (the first one they had constructed). Student S1 justified this choice by stating in line 456 that they should choose that one because it was "the most complete" definition, to which S3 and S4 agreed in lines 457 and 458, respectively. Later, S4 reminded S1 that he had previously said something, so S1 added that their chosen definition was also "the most specific". Therefore, the meta-level narratives of lines 456-458 informed us about the meta-rule that the PPTs first proposed to answer Question 9, which consisted of choosing the definition that had the greatest amount of information, even if some of it was unnecessary to a mathematician. In line 461, S1 introduced a different meta-rule: the definition had to be "the most specific". An analysis of this group's answers to previous questions shows that they understood that a definition for a particular solid was "specific" when that solid satisfied it, but the rest did not. Therefore, a definition can be "the most complete" and not "the most specific" and vice versa. However, the PPTs did not appear to realise that they had proposed two different meta-rules and accepted S4's proposal of writing both together "the most complete and specific" (line 462). Since both meta-rules can coexist in the same discourse, a commognitive conflict could not be inferred.

In vignette 3, we show a situation in which the existence of more than one meta-rule prompted a discussion about which criterion to employ when choosing a definition. Specifically, the three PPTs in group G2 had the following conversation:

251 S2: I would put both together, but that's me.
252 S3: Figure of six faces, twelve edges, eight vertices, six faces with square shape and its…

253 S1: That one, right?
254 S3: Yes.
255 S2: I would choose the definitions that we have written in Question 6.
[…]
258 S2: Why?
259 S3: Because it is more complete.
260 S2: Because it is describing with all the characteristics that the figure has.
261 S1: Because, if it tells you it has six faces, you don't know what those six faces look like.
262 S2: Because Question 1, what we are saying, the previous question what we are saying is that it has space and that it is a polyhedron with six faces, which can include…
263 S3: And also so that a child can identify the figure much better with the definition of Question 6 than with [the one of the] other question […]

The first line of this excerpt (line 251) is a narrative at meta level, since it shows the meta-rule that S2 initially proposed to answer Question 9: merging the two definitions that they had previously constructed for solid 1. S1 and S3, without explicitly stating the meta-rule that they would use to choose the best definition, proposed to choose the definition from Question 6 (lines 252-254). S2 accepted this decision in line 255, without any debate about why. To justify their choice, S3 stated that she preferred that definition because it was the most complete one (line 259), and then added that it was also the definition that would allow a child to "better" identify solid 1 (line 263). To support S3's justification in line 259, S2 said that the chosen definition was the one that described solid 1 with "all" its characteristics (line 260). In line 261, S1 supported this statement from S2, adding that, if they chose the definition in Question 5, information on what the faces of solid 1 were like would be missing. Therefore, the PPTs had made explicit the meta-rule that had guided their choice: choosing the definition that included the largest number of characteristics of solid 1, which they called "the most complete" definition.

In this case, the existence of two different meta-rules for choosing a definition does not seem to have led to a commognitive conflict. This is because both meta-rules coexist in the same discourse since the meta-rule proposed by S2 (merging both definitions) may have been motivated by the fact that S2 considered that the maximum amount of information should appear in a definition. S2 might have abandoned her meta-rule simply because she considered that it was not a valid answer to Question 9, which requires choosing between two definitions. In this case, it seems that a social norm (you have to "obey" what the question requests) has prevailed over the meta-rule that S2 originally wanted to use. In this work, social norms are considered to be those aspects of social interactions in class that become normative, such as the obligation to explain all reasoning and to try to understand the reasoning of others (Yackel et al., 2000).

DISCUSSION AND CONCLUSIONS

In this paper, three vignettes that summarise part of our findings have been presented to answer our research questions.

The first vignette shows two different meta-rules inferred from the PPTs' discursive activity when defining the geometric solids for the first time. The fact that such meta-rules exist in incommensurable discourses has allowed us to infer the existence of a commognitive conflict where some PPTs confuse the process of describing with that of defining.

In the other two vignettes, different meta-rules are inferred from the PPTs' discursive activity when choosing one definition from the two they had previously constructed for each of the solids. On the one hand, the second vignette shows how certain PPTs proposed two different meta-rules: choosing "the most complete" definition and choosing "the most specific" definition. In that vignette, the students decided to join both meta-rules and write "the most complete and specific one". On the other hand, the third vignette shows how certain PPTs proposed two different meta-rules in order to choose a definition: "describing with all the characteristics" and "merging both definitions". We highlight that, in both vignettes, since the meta-rules that had been identified can coexist in the same discourse, a commognitive conflict could not be inferred.

Therefore, we can answer our first research question: different meta-rules do not always lead to a commognitive conflict.

We focus on the first vignette to answer our second research question, concerning the reasons that give rise to a commognitive conflict and, if so, if it is resolved. We have inferred that a commognitive conflict may arise when some PPTs consider that defining a mathematical object is simply describing its properties while other PPTs do not. Some PPTs tried to answer a question on defining through a discursive activity typical of the discourse on describing. According to Tall (1991), one of the aspects that characterises the transition from elementary to advanced mathematical thinking is the transition from describing to defining. Specifically, the transition from a mathematical discourse on describing, in which there are only descriptions of mathematical objects but not their definitions, to a discourse on defining, which includes formal definitions of such objects. This transition involves a development of the first discourse at the meta level because the discourse on describing and the discourse on defining are incommensurable. Consequently, it is the PPTs' meta-level learning that enables them to progress from elementary to advanced mathematical thinking (from the discourse on describing to the discourse on defining) as a result of the resolution of commognitive conflicts.

Furthermore, in the second and third vignettes, the meta-rules inferred reveal that students prefer definitions with the maximum amount of information, even if some of this information is unnecessary. By connecting these results with the van Hiele levels of reasoning (Gutiérrez & Jaime, 1998), it could be said that the students in these two groups are in the second level of van Hiele due to their use of unnecessary and redundant information. Moreover, the existence of different meta-rules also proves that, even among students in the same van Hiele level 2 (Gutiérrez & Jaime, 1998), different discourses can be identified. These results are consistent with those of Wang and Kinzel (2014), who, through the theory of commognition, have identified the existence of different discourses of students who are at the same level 3. We also believe that it would be possible to characterise, in terms of the theory of commognition, the transition between Tall's (1991) Elementary Mathematical Thinking and Advanced Mathematical Thinking.

According to the proposal of Kobiela and Lehrer (2015) regarding "aspects of definitional practice" (p. 425), the characteristics of the students' activities related to the *selection of definitions* could be included as a new aspect for consideration. Moreover, it could also complement the process of reasoning in geometry related to definitions considering formulation, use, and selection of definitions (Gutiérrez & Jaime, 1998).

We acknowledge certain limitations of our work, such as the number of participants and the order and wording of certain questions in the data collection instrument, which may have induced the appearance of some specific meta-rules. In future work, we are interested in furthering the study of the relationships between the theory of commognition and the van Hiele levels when PPTs define geometric objects. Moreover, we would like to study how PPTs engage in both mathematical and pedagogical discourses when they answer questions about the selection of definitions for different purposes.

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REFERENCES

Avcu, R. (2022). Pre-service middle school mathematics teachers' personal concept definitions of special quadrilaterals. *Mathematics Education Research Journal*. https://doi.org/10.1007/s13394-022-00412-2

Biza, I. (2021). The discursive footprint of learning across mathematical domains: The case of the tangent line. *The Journal of Mathematical Behavior, 62,* Article 100870. https://doi.org/10.1016/j.jmathb.2021.100870

De Villiers, M. (1998). To teach definitions in geometry or teach to define? In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd International Conference for the Psychology of Mathematics Education* (Vol. 2, pp. 248-255). University of Stellenbosch.

De Villiers, M., Govender, R., & Patterson, N. (2009). Defining in geometry. In T. V. Craine & R. Rubinstein (Eds), *Understanding geometry for a changing world* (pp. 189-203). NCTM.

Fernández-León, A., Gavilán-Izquierdo, J. M., González-Regaña, A. J., Martín-Molina, V., & Toscano, R. (2021). Identifying routines in the discourse of undergraduate students when defining. *Mathematics Education Research Journal, 33*(2), 301-319. https://doi.org/10.1007/s13394-019-00301-1

Freudenthal, H. (1973). *Mathematics as an educational task.* Reidel. https://doi.org/10.1007/978-94-010-2903-2

Gallego-Sánchez, I., González, A., & Gavilán-Izquierdo, J. M. (2022). Analyzing pedagogical routines in the upper secondary school teacher's discourse using the commognitive approach. *International Journal of Instruction, 15*(3), 291-306.

Gavilán-Izquierdo, J. M., Martín-Molina, V., González-Regaña, A. J., Toscano, R., & Fernández-León, A. (2019). Cómo construyen definiciones matemáticas los estudiantes para maestro: Una aproximación sociocultural [How pre-service teachers construct mathematical definitions: A sociocultural approach]. In E. Badillo Jiménez, N. Climent Rodríguez, C. Fernández Verdú, and M. T. González Astudillo (Eds.), *Investigación sobre el profesor de matemáticas: Práctica de aula, conocimiento, competencia y desarrollo profesional* (pp. 135-155). Ediciones Universidad de Salamanca.

González-Regaña, A., Martín-Molina, V., Toscano, R., Fernández-León, A., & Gavilán-Izquierdo, J. M. (2021). El discurso de estudiantes para maestros cuando describen y definen cuerpos geométricos [Pre-service teachers' discourse when describing and defining geometric solids]. *Enseñanza de las Ciencias, 39*(1), 81-97. https://doi.org/10.5565/rev/ensciencias.3039

Gutiérrez, A., & Jaime, A. (1998). On the assessment of the van Hiele levels of reasoning. *Focus on Learning Problems in Mathematics, 20*(2/3), 27-46.

Ioannou, M. (2018). Commognitive analysis of undergraduate mathematics students' first encounter with the subgroup test. *Mathematics Education Research Journal, 30*(2), 117-142. https://doi.org/10.1007/s13394-017-0222-6

Kobiela, M., & Lehrer, R. (2015). The codevelopment of mathematical concepts and the practice of defining. *Journal for Research in Mathematics Education, 46*(4), 423-454. http://doi.org/10.5951/jresematheduc.46.4.0423

Kusumah, Y. S., Kustiawati, D., & Herman, T. (2020). The effect of GeoGebra in three-dimensional geometry learning on students' mathematical communication ability. *International Journal of Instruction, 13*(2), 895-908. https://doi.org/10.29333/iji.2020.13260a

Lavie, I., Steiner, A., & Sfard, A. (2019). Routines we live by: From ritual to exploration. *Educational Studies in Mathematics, 101*(2), 153-176. https://doi.org/10.1007/s10649-018-9817-4

Leikin, R. & Zazkis, R. (2010). On the content-dependence of prospective teachers' knowledge: A case of exemplifying definitions. *International Journal of Mathematical*

Education in Science and Technology, 41(4), 451-466. https://doi.org/10.1080/00207391003605189

Mariotti, M. A., & Fischbein, E. (1997). Defining in classroom activities. *Educational Studies in Mathematics, 34*(3), 219-248. https://doi.org/10.1023/A:1002985109323

Miller, S. M. (2018). An analysis of the form and content of quadrilateral definitions composed by novice pre-service teachers. *The Journal of Mathematical Behavior, 50*, 142-154. https://doi.org/10.1016/j.jmathb.2018.02.006

Nachlieli, T., & Heyd-Metzuyanim, E. (2022). Commognitive conflicts as a learning mechanism towards explorative pedagogical discourse. *Journal of Mathematics Teacher Education, 25,* 347-369. https://doi.org/10.1007/s10857-021-09495-3

Nachlieli, T., & Tabach, M. (2022). Classroom learning as a deritualization process: The case of prospective teachers learning to solve arithmetic questions. *The Journal of Mathematical Behavior, 65,* 1-15. https://doi.org/10.1016/j.jmathb.2021.100930

Nardi, E., Ryve, A., Stadler, E., & Viirman, O. (2014). Commognitive analyses of the learning and teaching of mathematics at university level: The case of discursive shifts in the study of Calculus. *Research in Mathematics Education, 16*(2), 182-198. https://doi.org/10.1080/14794802.2014.918338

Rasmussen, C., Zandieh, M., King, K., & Teppo, A. (2005). Advancing mathematical activity: A practice-oriented view of advanced mathematical thinking. *Mathematical Thinking and Learning, 7*(1), 51-73. https://doi.org/10.1207/s15327833mtl0701_4

Sánchez, V., & García, M. (2008). What to teach and how to teach it: Dilemmas in primary mathematics teacher education. In B. Jaworski & T. Wood (Eds.), *International Handbook of Mathematics Teacher Education. Volume 4: The mathematics teacher educator as a developing professional* (pp. 281-297). Sense Publishers.

Sánchez, V., & García, M. (2014). Socio-mathematical and mathematical norms related to definition in pre-service primary teachers' discourse. *Educational Studies in Mathematics, 85*(2), 305-320. https://doi.org/10.1007/s10649-013-9516-0

Schüler-Meyer, A. (2020). Mathematical routines in transition: Facilitating students' defining and proving of sequence convergence. *Teaching Mathematics and its Applications: An International Journal of the IMA, 39*(4), 237-247. https://doi.org/10.1093/teamat/hrz019

Sfard, A. (2008). *Thinking as communicating. Human development, the growth of discourse, and mathematizing.* Cambridge University Press. https://doi.org/10.1017/CBO9780511499944

Sfard, A. (2021a). Bewitched by language: Questions on language for mathematics education research. In N. Planas, C. Morgan, & M. Schütte (Eds.), *Classroom research on mathematics and language: Seeing learners and teachers differently* (pp. 41-59). Routledge.

Sfard, A. (2021b). Taming fantastic beasts of mathematics: Struggling with incommensurability. *International Journal of Research in Undergraduate Mathematics Education.* https://doi.org/10.1007/s40753-021-00156-7

Tabach, M., & Nachlieli, T. (2015). Classroom engagement towards using definitions for developing mathematical objects: The case of function. *Educational Studies in Mathematics, 90*(2), 163-187. https://doi.org/10.1007/s10649-015-9624-0

Tall, D. (Ed.). (1991). *Advanced Mathematical Thinking.* Kluwer. https://doi.org/10.1007/0-306-47203-1

Thoma, A., & Nardi, E. (2018). Transition from school to university mathematics: Manifestations of unresolved commognitive conflict in first year students' examination scripts. *International Journal of Research in Undergraduate Mathematics Education, 4*(1), 161-180. https://doi.org/10.1007/s40753-017-0064-3

Viirman, O., & Nardi, E. (2019). Negotiating different disciplinary discourses: Biology students' ritualized and exploratory participation in mathematical modeling activities. *Educational Studies in Mathematics, 101*(2), 233-252. https://doi.org/10.1007/s10649-018-9861-0

Wang, S., & Kinzel, K. (2014). How do they know it is a parallelogram? Analysing geometric discourse at van Hiele level 3. *Research in Mathematics Education, 16*(3), 288-305. https://doi.org/10.1080/14794802.2014.933711

Yackel, E., Rasmussen, C., & King, K. (2000). Social and sociomathematical norms in an advanced undergraduate mathematics course. *The Journal of Mathematical Behavior, 19*(3), 275-287. https://doi.org/10.1016/S0732-3123(00)00051-1

Zaslavsky, O., & Shir, K. (2005). Students' conceptions of a mathematical definition. *Journal for Research in Mathematics Education, 36*(4), 317-347.

Zazkis, R., & Leikin, R. (2008). Exemplifying definitions: A case of a square. *Educational Studies in Mathematics, 69*(2), 131-148. https://doi.org/10.1007/s10649-008-9131-7