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Exploring the Primary School Teachers' Reasoning and Individuals with Limited Education When Solving False Proportionality Problems

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In this study employing a qualitative approach, we explore and compare the mathematical reasoning of active primary school teachers and individuals with limited education when tackling problems involving false proportionality and lack of authenticity. The research is justified by the importance of understanding how these issues are addressed in various educational groups and how teaching strategies in this area can be enhanced. An individual questionnaire containing five problems was administered to five participants in each group. The results indicate that teachers tend to more frequently succumb to the illusion of linearity, incorrectly applying linear relationships to non-proportional problems compared to individuals with lower educational attainment. This suggests a greater influence of algorithmic approaches and clauses of the didactic contract on their mathematical reasoning. This tendency could be attributed to a greater exposure to algorithmic approaches during the formative stages, as well as certain clauses in the experimental contract. Conversely, individuals with limited education approached problems with greater flexibility, linking their responses to everyday experiences. The study concludes that it is imperative to implement educational strategies aimed at overcoming limitations such as the illusion of linearity and fostering meaningful problem-solving skills from the early stages. It is recommended to developteacher training programs that promote a more contextualized and flexible approach to mathematics teaching, as well as the include contextualized and meaningful problems in the school curriculum.

Keywords: illusion of linearity, proportionality, primary school teachers, limited education, authenticity

INTRODUCTION

Proportional reasoning is an essential concept in mathematics that plays a crucial role in students' academic development (Lesh et al., 1988). It is defined as the ability to establish multiplicative relationships between quantities, requiring an understanding of

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the covariance between quantities—when one quantity varies, the second also changes in response (Copur-Gencturk et al., 2023).

Freudenthal defines ratios as multiplicative relationships between two quantities and proportional relationships between sets of ratios that make a true proportion within mathematical contexts.

Proportionality holds a 'unique duality' in mathematics (Lesh et al., 1988). It is a fundamental understanding, yet it represents an advanced comprehension that significantly influences students' mathematical development. Many students struggle to develop a deep understanding of proportionality (De Bock et al., 2007), partially due to how it is taught in the curriculum, emphasizing mechanical resolution over contextual analysis. One obstacle is the 'illusion of linearity', leading to the incorrect application of linear properties, underscoring the need to distinguish linear from non-linear situations in proportional reasoning.

This work focuses on exploring the reasoning of primary school teachers compared to individuals with limited education when confronted with exercises designed to induce the illusion of linearity. By examining the reasoning processes of these two groups, we aim to shed light on the factors influencing their approaches to mathematical challenges. In this article, we define individuals with 'limited education' as those who have completed at most the sixth grade of primary school, typically achieved around the age of 12 in Mexico. This definition allows us to compare these individuals' problem-solving strategies with those of primary school teachers.

The exercises in the applied questionnaire also have a low authenticity factor, implying they have minimal probability of occurring in real life. Thus, it seems they can only be solved within a school context and mathematically. This decision was made to observe how participants approach purely mathematical problems, highlighting the importance of distinguishing between mathematical concepts and their practical applications. The study's objective is to compare the reasoning of both groups when solving these exercises. Therefore, the research question we've defined is: *What reasoning do individuals with limited education and primary school teachers employ when responding to a questionnaire on false proportionality exercises and inauthentic problems?*

Overall, this study addresses the gap in understanding how different levels of education impact individuals' mathematical reasoning, particularly in the context of proportional reasoning and the illusion of linearity. By providing insights into the reasoning processes of primary school teachers and individuals with limited education, we contribute to the ongoing discourse on effective mathematics education strategies.

We anticipated that, for the design and application of proposed exercises on false proportionality and inauthenticity, individuals with limited education would be less likely to fall into the illusion of linearity. This expectation arises from their lower adherence to clauses established by the 'didactic contract' and their familiarity with solving problems outside the classroom compared to primary teachers.

Literature Review

The illusion of linearity or pseudo-proportionality is a phenomenon deeply ingrained in mathematics education. Linear models, such as the rule of three, are predominant in the minds of students from an early age through higher education due to their simplicity and versatility. This familiarity with linearity can lead to a misperception that these models are universally applicable to any seemingly proportional situation (Rizos & Foykas, 2023). Resistance to this illusion has been investigated, both through methods to reduce error and when solving problems with direct measurements of area. It was observed that there is no higher success rate compared to problems solved with indirect measurements (Van Dooren et al., 2003).

This linear or proportional model is a key concept that receives significant attention in mathematics and science education. However, several mathematics researchers and educators (e.g., Freudenthal, 1983) have cautioned that students may develop a tendency to think that every numerical relationship is linear. Despite these illustrations of students' inclination towards inappropriate proportional reasoning, systematic analyses of the illusion of linearity are surprisingly scarce (Van Dooren et al., 2003).

Rizos & Foykas (2023) underscore the fundamental importance of proportionality, not only in everyday life but also in scientific and mathematical disciplines. In their analysis, they highlight the concept of proportional reasoning, largely equated to linear reasoning, which is fundamental in mathematics education and involves expressions of relationships such as $f(ax) = af(x)$. However, they emphasize that the current educational methodology adheres to a traditional model, where students tend to mechanically copy without deepening their understanding of mathematical concepts. This practice limits the genuine comprehension of proportionality and its applicability in everyday situations, making it challenging to identify phenomena in both mathematical and daily life problems.

Studies also point to the existence of common intuitive rules that influence students' responses to nonlinear situations in mathematics and science. One of these rules, identified as 'more A - more B', posits that when students compare two objects with respect to a salient quantity A (A1 > A2), they intuitively reason about another pair of quantities B (B1 > B2) (De Bock et al., 2002).

This rule is related to the tendency to apply multiplication to non-linear situations, represented by the rule 'k times A equals k times B', a quantification directly derived from the rule 'more A - more B' (De Bock et al., 2002). Furthermore, it has been exemplarily reported in students of different ages and in various fields of mathematical and scientific education, such as elementary arithmetic, algebra, and physics, and has been systematically studied in geometry.

Teaching the correct algorithm could be a helpful tactic, but it may not be enough to overcome the inclination toward linear thinking ingrained in the intuitive understanding of proportionality (Miszaniec, 2016). This discovery indicates that simply being exposed to the correct procedure might not be sufficient to dispel the illusion of linearity, as this perception could be deeply ingrained in individuals' intuitive comprehension of proportional relationships. This emphasizes the necessity for broader and more contextualized educational strategies that not only instruct on the correct algorithm but also tackle and question the underlying intuitions that could promote linear thinking in proportional situations.

In a study involving individual interviews (De Bock et al., 2002), information was gathered on problem-solving processes and the explanatory factors underlying the tendency to produce linear responses. First, the results showed that most students spontaneously and almost intuitively use the linear model, while some students are convinced that linear functions are applicable 'everywhere'. Furthermore, many students exhibit inappropriate habits, beliefs, and attitudes toward problem-solving in mathematics, leading to stereotypical and superficial mathematical modeling.

Amaro et al. (2019) have identified an opportunity to improve the treatment of this concept in education. They suggest that both authors and educational institutions consider the need to propose activities that help students overcome the illusion of linearity, a crucial aspect for a deeper understanding of proportionality. The teaching-learning process presents a challenge for both teachers and students, as it is often difficult to effectively transmit this knowledge from the former to the latter. Additionally, there is the complexity of distinguishing between proportional and non-proportional situations, for which it is necessary to promote more meaningful and less algorithmic knowledge.

The excessive use of mechanical reasoning to solve proportional problems without logical thinking employed in the resolution is the main source of the illusion of linearity. Van Dooren et al. (2006) have documented the abusive application of linearity in the teaching of proportionality. This abuse is, in part, due to the introduction of proportionality as an isolated notion without considering its appropriateness for a particular situation. Proportionality problems arise in a context that forces a simplistic technical management, where students only have to determine if 'more is more; unless is less.' Freudenthal used the term *linear* as a synonym for *proportional* referring to relationships graphically represented by a straight line passing through the origin.

The linear illusion in mathematics is a phenomenon of interest to many authors. As noted by Lesh et al. (1988), because proportionality is one of the most basic higher-order understandings, on the one hand, and one of the most advanced basic understandings, on the other, proportional reasoning skills play an important role in students' mathematical development.

Linearity manifests itself throughout a person's entire school career, as suggested by De Bock et al. (2007). It covers the entire mathematical edifice, from the idea of measuring magnitudes, the concept of proportions, the learning of fractions (Wijaya, 2017), and the application of the 'rule of three' in elementary school, to linear algebra and the use of linear models in calculus and statistics in high school. It extends even to abstraction in a vector space sense in higher education. Linear relationships serve as essential models that help understand and address various problematic situations, both practical and theoretical in mathematics, which, in turn, solve problems in everyday life; for this reason, they receive significant attention in contemporary mathematics education (De

Bock et al., 2007). In line with these approaches, to have a greater learning impact, the activities suggested and implemented by teachers must have a sense of coherence with reality through authentic situations.

In a study by Aguerrea et al. (2020), persistent errors in mathematical concepts and procedures, especially the incorrect application of linearity, were identified. The study included 42 mathematics education students during the 2019-2020 period, with tests administered at the beginning and after a semester of training, followed by workshops to address the errors. Despite the training received, many students persisted in making errors in certain concepts, emphasizing the need for more effective teaching approaches.

Similarly, Duma (2021) explored the widespread impact of linearity in daily decision-making, revealing its excessive or unfounded application. The study investigated different scenarios where misconceptions about linearity resulted in flawed decisions, such as in interpreting MPG (Miles per Gallon) and MPH (Miles per Hour) indicators. Through survey research, Duma illuminated the prevalence of linear thinking in everyday decision-making and uggested psychological mechanisms and policy interventions to mitigate these errors. These findings highlight the importance of addressing misconceptions about linearity not only in mathematical education but also in broader contexts to improve decision-making processes.

The gap between solving mathematical problems and their relationship with reality demonstrates the limitations of current teaching methods. Mathematics education faces the challenge of teaching skills applicable to real-world problems, not just solving problems in the classroom (Khoshaim, 2020; Wisenöcker et al., 2023). In class, students often prioritize 'correct' numerical answers without considering context, due to school rules that emphasize accuracy over comprehension (Yackel & Cobb, 1996). This hinders a deep understanding of the real applicability of mathematics (Wisenöcker et al., 2023).

In schools, the stereotype of solving problems by selecting operations without considering their relevance is a barrier to broad mathematical understanding. Transforming teaching is required to close the gap between mathematical theory and everyday life (Wisenöcker et al., 2023). Incorporating realistic word problems seeks to improve mathematical understanding and link it to meaningful applications in everyday life (Burton, 1993; Niss, 1992).

The need to cultivate creative thinking and avoid the illusion of linearity is essential to developing students' critical thinking, since the extreme application of linear patterns can limit their ability to solve authentic, everyday problems in innovative and effective ways (Rizos and Foykas, 2023).

On the other hand, the 'didactic contract' (Brousseau, 1980) represents a tacit agreement between teachers and students that impacts mathematical learning in the classroom (D'Amore, 2006). Students assume the teacher's expectations, influencing the teachers' problem-solving. This dynamic, based on institutional perceptions and beliefs, limits creative exploration and deep understanding, leading to expected responses.

D'Amore (2006) exemplifies how the experimental contract impacts problem-solving. Students respond to perceived expectations of the school and subject, influencing their approach. For example, they may restrict answers and adopt formal language, assuming that the questions involve evaluation. They also look for numerical answers, even if the problem does not require it, based on previous assumptions. Furthermore, the interesting question of how students approach unsolvable problems is presented, which poses a gap between their expectations and the reality of the problem. This situation challenges your usual assumptions about the feasibility of questions, ultimately leading to a reevaluation of your perspective and approach to problem-solving.

METHOD

Pretest

Study Design

The study conducted follows a qualitative approach with a descriptive scope, as we focus on analyzing and describing the responses of the subjects, as well as how they answered each of the approaches outlined in the questionnaire.

Participants

Participants for the pretest were chosen in a similar manner to those in the main study. Individuals with characteristics resembling the main study's target group, who also had limited education and fell within the same age range, were recruited. A total of 3 participants from each group completed the pretest.

Pretest Instrument

The pretest included a series of items aimed at evaluating participants' proficiency in recognizing issues related to proportionality and authenticity. Pretest items were adapted from the main instrument to ensure their relevance and comprehensibility to participants. Changes were implemented to certain items that caused confusion during the pilot process. The details of the pretest instrument are provided in Annex 2.

Pretest Procedure

Participants completed the pretest instrument in a controlled environment, following the same instructions and conditions that would be used in the main study. Their responses were recorded, and the validity and reliability of the instrument were analyzed based on the results obtained. Three individuals from each participant group (limited education and primary school teachers) were surveyed. They were label as I (Illusion) if they fell for the linearity illusion trap and NI (No Illusion) if they did not, regardless of the mathematical correctness of their responses. For simplicity, participants were referred to as P1, P2, and P3. Results for individuals with limited education are presented in Table 1, while those for primary school teachers are shown in Table 2.

Table 1
Results obtained in primary teachers

	Item	T1	T2	T3
	1	I	NI	I
True proportionality	2	NI	NI	NI
	3	NI	I	I
	4	I	I	I
	5	NI	NI	I

Table 2
Results obtained in individuals with limited education

	Item	P1	P2	P3
	1	NI	I	I
True proportionality	2	NI	NI	I
	3	NI	NI	I
	4	NI	NI	I
	5	NI	I	I

Bar charts (Figure 1 and Figure 2) were created to visualize the results for individuals with limited education and primary school teachers, respectively, to see how many exercises the participants were susceptible to the linearity illusion and how many they were not. In this pretest, it was observed that primary school teachers are more likely to succumb to the linearity illusion than individuals with limited education.

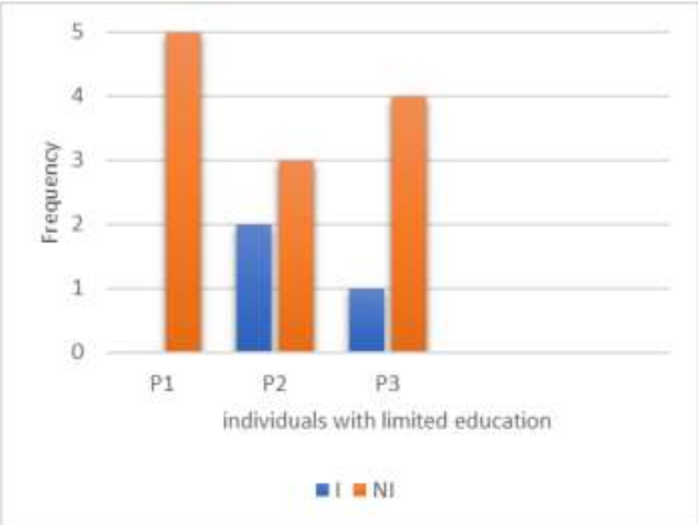


Figure 1
Participants with limited education

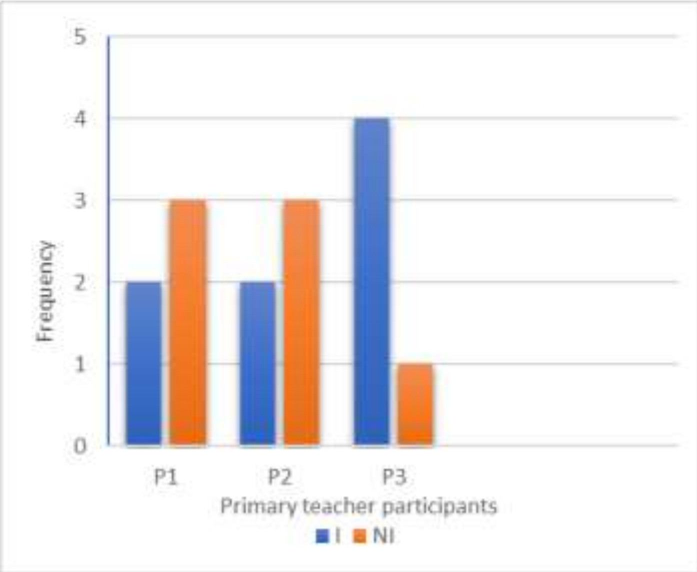


Figure 2
Primary teacher participants

Pretest Results Analysis

The pretest data were analyzed to evaluate the clarity and relevance of the items and to pinpoint potential areas of confusion or ambiguity. Participant feedback and suggestions were considered to refine the instrument before its use in the main study. For example, for item 3, participant P1 responded that the answer relied on the height of the window, which was not provided in the exercise despite mentioning the building's height of 100 m. This situation was similar for other participants.

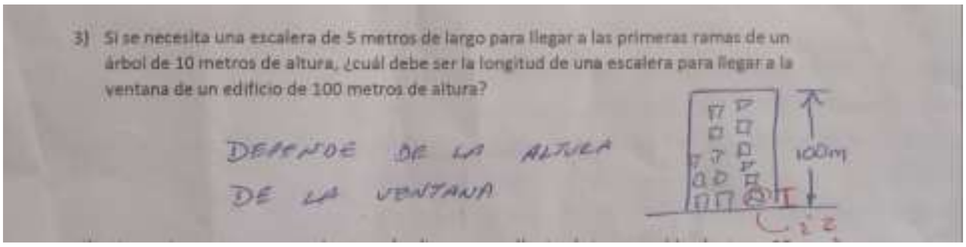


Figure 3
Participant 1 (P1), primary teacher participant

It was decided to change this item, as well as item 1, as it was found that it was relatively easy to identify the linearity illusion in that exercise, which alerted participants that the following exercises probably contained that trap.

Implementation of the Main Instrument

After completing the pretest and making necessary adjustments to the instrument, the main study was carried out with the selected participants. The final instrument was administered using the same procedures and conditions as those employed during the pretest.

Participants

The participants in this research consisted of two groups: individuals with *limited education* and, on the other hand, in-service primary school teachers. In this study, individuals with limited education were defined as those who had completed a maximum of the sixth grade of Primary Education, typically around the ages of 11 to 12 in Mexico. For the purposes of this study, it is considered that the lower the education level of the person surveyed, the better it helps to contrast their responses against those of primary school teachers. Therefore, the selection criterion was to survey individuals who had reached the lowest school grade among all those considered. The decision to include the second group was solely based on their active status in teaching service, and the selection was carried out through invitation only.

Instrument

The instrument applied consisted of five problems designed with the purpose of identifying whether the subjects were capable of detecting false proportionality and lack of authenticity. It should be noted that within the instrument, a single problem of true proportionality was included, with the purpose that if any participant realized the trap in the approaches, they would tend to solve them all in the same way. Furthermore, all problems were presented as missing value problems.

In this type of problem, three numbers (a, b, c), are given and the problem solver is asked to determine an unknown number, x. In a proportional missing value problem, the x unknown is the solution to an equation of the form a/b=c/x. (De Bock et al., 2002). The instrument is found in Appendix of this document.

FINDINGS

Five individuals from each group of participants (those with limited education and primary school teachers) were surveyed. The acronym I (Illusion) was used if the person fell into the illusion of linearity trap, and NI (No Illusion) if, on the contrary, they did not fall, regardless of whether the results are correct or not from a mathematical point of view. For simplicity, the participants have been labeled as P1, P2, P3, P4, P5, T1, T2, T3, T4, T5 for individuals with limited education and primary school teachers, respectively. Table 1 displays the results obtained from the responses of individuals with limited education.

Table 2
Results obtained in primary teachers

	Item	T1	T2	T3	T4	T5
	1	I	I	I	I	I
True proportionality	2	I	I	I	I	I
	3	NI	I	I	I	I
	4	NI	I	I	NI	I
	5	NI	I	NI	NI	NI

Table 3
Results obtained in individuals with limited education

	Item	P1	P2	P3	P4	P5
	1	I	NI	I	NI	I
True proportionality	2	I	I	I	I	I
	3	I	NI	NI	NI	NI
	4	I	NI	NI	NI	NI
	5	NI	NI	NI	NI	NI

In addition, there are bar graphs (Figure 4 and Figure 5) to visualize the results of individuals with limited education and primary school teachers, respectively, in order to have a count and observe how many of the applied exercises led the participants to fall into the illusion of linearity and how many did not in the instrument. It is observed that, in fact, the assumption that primary school teachers would fall into this illusion of linearity more than individuals with limited education is confirmed.

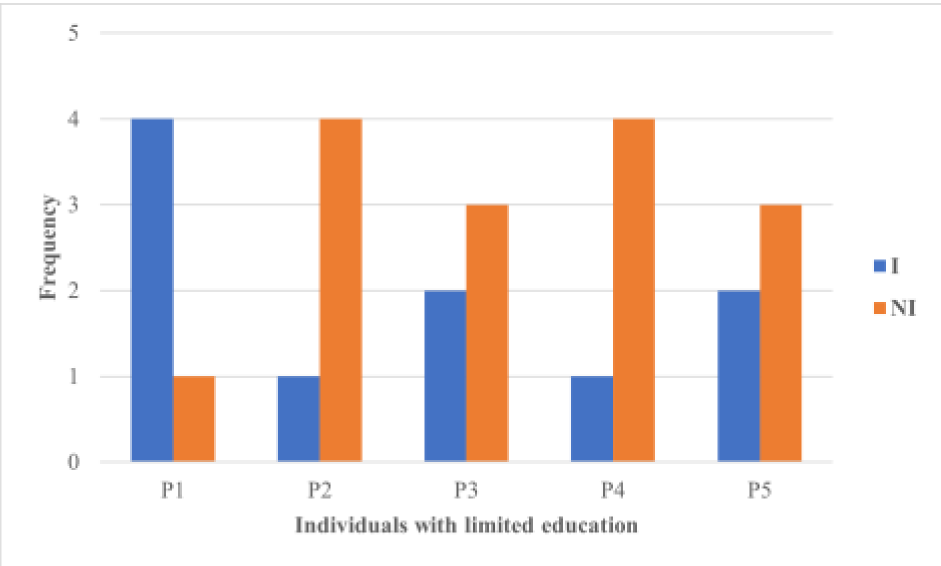


Figure 4
Participants with limited education

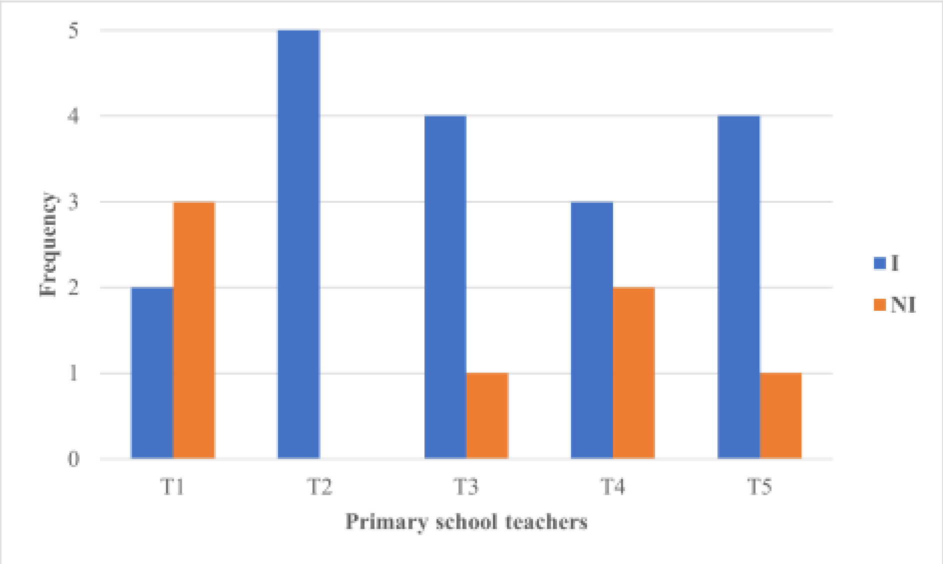


Figure 5
Primary teacher participants

Below are some of the most interesting results that, based on what was stated in this research, better represent the illusion of linearity and the didactic contract, whether or not they presented mentioned illusion. Each of the items has been named A1, A2, A3, A4, and A5, according to the order in which they appear in the applied instrument.

Results on People with limited education

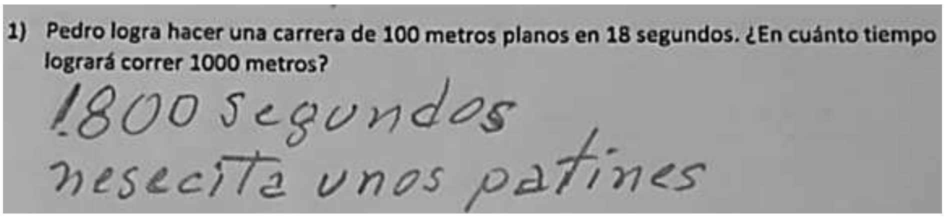
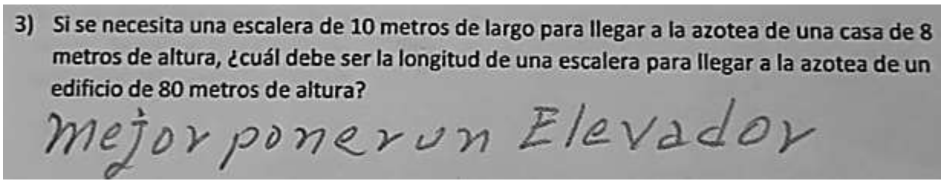


Figure 6
Participant 2 (P2), Item 1 (A1)

Initially, subject P2 responded that Pedro would take 1800 seconds to run 1000 meters, applying procedural mathematical reasoning. This shows that initially, he operated under the experimental contract, assuming that the exercises had to be answered formally. However, upon noticing inconsistencies in later problems, he reevaluated his initial solution, concluding that it did not seem realistic. Therefore, he opined, in a more creative way, that Pedro would need some skates to achieve that time. In this way, he breaks with the experimental contract and provides an answer that he considers more in line with the context presented.

The above is related to the notion that students must believe that their solutions will be judged according to real-world requirements, not according to teacher expectations (Verschaffel et al., 2009). Thus, for example, subject P2 demonstrated the ability to consider the problem scenario to solve it meaningfully.

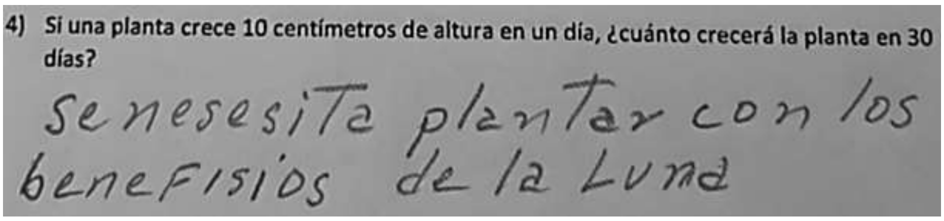

3) Si se necesita una escalera de 10 metros de largo para llegar a la azotea de una casa de 8 metros de altura, ¿cuál debe ser la longitud de una escalera para llegar a la azotea de un edificio de 80 metros de altura?

mejor poner un Elevador

Figure 7
Participant 2 (P2), Item 3 (A3)

This approach posed a greater challenge for the participant, as up to that point, he had assumed the researchers' expectation of necessarily reaching a formal solution through a mathematical procedure, evoking the experimental contract (Brousseau, 1980). Faced with the desperation of not finding said 'correct' procedure, subject P2 thought in a more practical way, suggesting that it was better to install an elevator to solve the situation at hand.

In this manner, in his eagerness to comply with what he perceived as the formal requirements of the problem, subject P2 finally demonstrates the ability to consider the context of the approach, leaving aside the strictly algorithmic approach. This flexibility to incorporate realistic situational aspects evidences an advancement in their understanding of the broader applicability of mathematics.


4) Si una planta crece 10 centímetros de altura en un día, ¿cuánto crecerá la planta en 30 días?

Se nesesita plantar con los benefisios de la Luna

Figure 8
Participant 2 (P2), Item 4 (A4)

Although subject P2 was less influenced by the experimental contract, it still persisted. He was then encouraged to respond more deeply and creatively. After a few minutes of analysis, he recalled his grandmother's beliefs about planting crops in specific lunar seasons to optimize their growth. Thus, subject P2 responded that it would be necessary to '*plant with the benefits of the moon*', demonstrating an attachment to ancestral knowledge of his community and using it to validly solve the problem according to his situational perspective.

However, reviews of scientific literature do not show reliable causal relationships between lunar phases and plant physiology that support these practices (Mayoral et al., 2020). Despite this, P2's response denotes consideration of his sociocultural context by connecting mathematical solutions with previous knowledge and experiences, bridging gaps between algorithmic formality and contextualized reasoning.

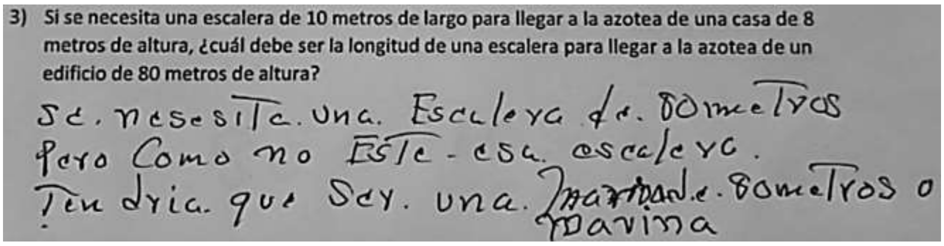


Figure 9
Participant 3 (P3), Item 3 (A3)

Like in the case of P2, individual P3 initially attempted to restrict his response to formal language, assuming that the question involved evaluation. After solving it mathematically, he linked the result with realistic applications (Burton, 1993; Niss, 1992). Thus, he proposed that an 80-meter ladder would be needed, but since it did not exist, an 80-meter marine ladder could be used. This solution piqued our curiosity. When asked what marine stairs were, he explained that it referred to emergency stairs built into tall buildings, lending real logic to his answer obtained previously by formal procedures.

In this way, subject P3 demonstrates the ability to connect mathematical answers with practical meanings, starting formally by meeting perceived expectations, but also bridging gaps between learned theory and authentic applications.

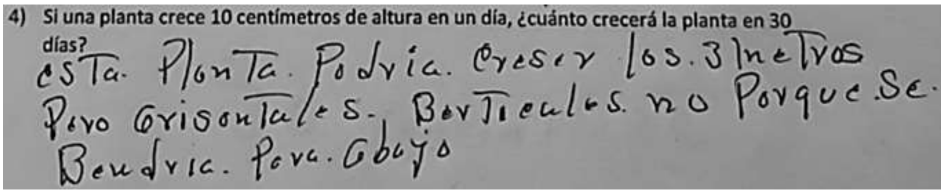
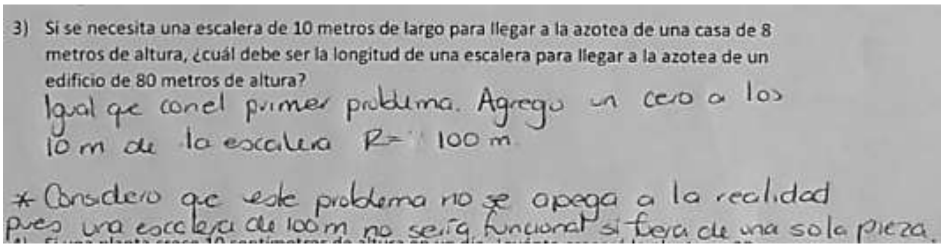


Figure 10
Participant 4 (P4), Item 4 (A4)

For the fourth approach, initially, subject P4 arrived at a mathematical solution. However, upon analyzing this response and drawing on previous experience, he reasoned that such a plant could perhaps exist, but only with horizontal growth like vines. According to subject P4, vertical growth would cause the weight to make it fall.

This denotes an effort to link the formal result with practical restrictions, bridging gaps between theory and reality (Wisenöcker et al., 2023). Although subject P4 met perceived expectations, he also connected his response to meaningful applications (Burton, 1993; Niss, 1992). Thus, progress is evident in P4 in understanding the need to complement what is formally learned with situational reasoning to improve the broader applicability of mathematics.

Results on Primary Teachers


3) Si se necesita una escalera de 10 metros de largo para llegar a la azotea de una casa de 8 metros de altura, ¿cuál debe ser la longitud de una escalera para llegar a la azotea de un edificio de 80 metros de altura?

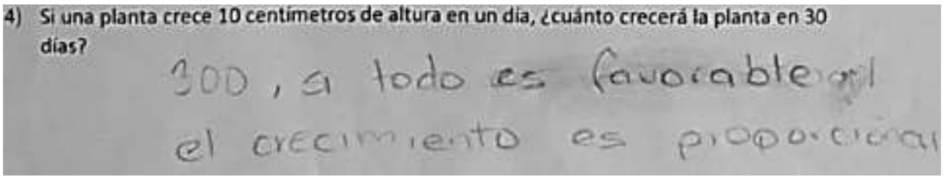
Igual qe conel primer problema. Agrego un cero a los 10 m de la escalera R= 100 m.

* Considero qe este problema no se apega a la realidad pues una escalera de 100 m no sería funcional si fuera de una sola pieza.

Figure 11
Participant 1 (T1), Item 3 (P3)

Although the teachers mostly evidenced the illusion of linearity and experimental contract, some responses denoted some resistance. Initially, teacher T1 replicated this bias by adding a zero to obtain a 100-meter ladder, but later recognized that this result does not correspond to reality since such a ladder would not be functional.

This reevaluation exemplifies an 'experimental contract', with more open communication to understand tacit and explicit expectations in problem-solving (D'Amore, 2006). Teacher T1 moves from stereotyped thinking to reasoning about practical constraints of the situation, demonstrating greater awareness of the need for a connection between theory and real applicability.


4) Si una planta crece 10 centímetros de altura en un día, ¿cuánto crecerá la planta en 30 días?

300, si todo es favorable

el crecimiento es proporcional

Figure 12
Participant 3 (T3), Item 4 (A4)

Teacher T3 responded by applying proportional reasoning, stating that the growth would be 300 centimeters if everything is favorable. This denotes persistence of the illusion of linearity. However, the doubt expressed in '*if everything is favorable*' could indicate recognition that the situation would only be real under certain ideal conditions. Thus, although teacher T3 prioritizes a precise numerical result, it also evidences some realistic thinking.

This contradicts statements that teachers tend to focus on formal responses, neglecting the context (Yackel & Cobb, 1996). Therefore, teacher T3 shows progress in favoring situational reasoning, although it is still conditioned to obtaining a theoretically correct solution.

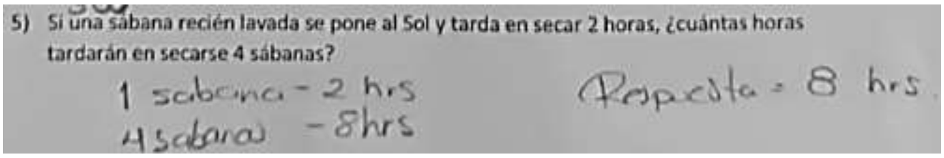


Figure 13
Participant 5 (T5), Item 5 (A5)

Teacher T5's response, who completely fell into the illusion of linearity and ignored the context of the exercise, reminds us of the proportionality rule 'more A - more B'. According to this rule, when students compare two objects that differ by a certain salient quantity A (A1 > A2), they intuitively defend another quantity B such that B1 > B2. However, those who fall into the linearity trap tend to apply the 'k times A - k times B' rule, which can be considered a direct multiplicative quantification of the 'more A - more B' rule. This finding highlights the importance of helping subjects understand the role of context and nonlinearity in solving proportionality problems (De Bock et al., 2002).

DISCUSSION

The results of this research show that the illusion of linearity is more prevalent in primary school teachers than in people with a lower education level. Of the five teachers surveyed, four of them showed this bias in most of the problems, while in the group with less education, only one of the participants systematically fell into this trap.

This difference could be due to the fact that, although linear reasoning emerges early at a developmental level, primary instruction reinforces it by focusing on linear relationships, making it more intuitive than non-linear patterns (De Bock et al., 2002). Therefore, nonlinear problems represent incongruent situations where the linear biased response is based on heuristics, and the nonlinear one is analytically and normatively correct (Putarek & Vlahović-Štetić, 2019).

Additionally, participants may apply stereotypical elementary school steps, focusing on obtaining an exact numerical result without realistic considerations. This leads to mathematically correct solutions, but not representative of the modeled situation (Wisenöcker et al., 2023). The over-reliance on intuitive linear schemes, reinforced by primary education, persists more in teachers and can bias their conclusions in non-linear situations. More research is needed to better understand this phenomenon.

Additionally, observations made during the pretest indicated that participants tended to rely on procedural reasoning, possibly because they perceived the interview as a school setting, triggering the didactic contract in which a correct response is anticipated. Initially, all participants in this study tended to use procedural reasoning, perhaps because they viewed the interview as a school environment, activating the didactic contract where a correct answer is expected (D'Amore, 2006). For example, individual P2 correctly solved the first two exercises procedurally. Unable to solve the third, he desperately appealed to the inauthenticity of the problem and proposed adding an elevator. For the fourth exercise, he resorted to a belief of his grandmother, recognizing that the situation posed could occur, but only '*with the benefits of the moon*'.

This type of intuitive reasoning based on prior knowledge is known as Type 1 processing (Stanovich & Toplak, 2012). Type 1 processing is fast, automatic, and relies on heuristics, while Type 2 is slow, requires cognitive effort, and relies on working memory. Among the teachers, although the majority showed the illusion of linearity, some responses were interesting. For example, teacher P3 supported his answer with mathematical reasoning but admitted that it would only be possible under ideal conditions.

Research shows that in incongruent problems, physiological intuitions ('hunches') are activated, indicating a possible error, although explicit responses remain biased (De Neys et al., 2010). In these cases, intuitive Type 1 processing generates a correct response, without the need for conscious Type 2 analysis, resulting in both processing being congruent and logically correct. However, in other incongruent problems, the automatic Type 1 response conflicts with the Type 2 logical analysis. Here, the analytical processing must override and inhibit the biased intuitive response to reach the normatively correct solution (Putarek & Vlahović-Štetić, 2019).

When comparing our findings with previous research, such as the study by Aguerre et al., (2020) on errors in mathematical concepts and procedures, we observe similarities and differences. Although both reveal the persistence of errors, our focus on the illusion of linearity suggests that it is more common among elementary school teachers than individuals with limited education. Additionally, while we used contextualized tasks, the previous study utilized collaborative workshops. These differences emphasize the need for more effective teaching strategies.

The findings of this study are relevant to the field of mathematics education, as they provide a deeper understanding of cognitive biases and thinking tendencies in solving proportionality problems. This can inform the creation of more effective teaching strategies that address the illusion of linearity from an early stage of learning. Furthermore, it underscores the importance of teacher training in promoting more reflective and flexible mathematical thinking.

The study design and data collection procedures are described in detail in the methods section, enabling other researchers to easily replicate the study in similar settings. Researchers are encouraged to use the same methodological approach to validate and extend the findings presented here, which could further contribute to the existing knowledge in the field.

CONCLUSIONS

This study explored and compared the reasoning used by individuals with limited education and in-service primary school teachers when solving a series of mathematical problems with false proportionality and lack of authenticity. The results showed that, in general, teachers tend to more frequently fall into the illusion of linearity than individuals with limited education, applying linear relationships incorrectly to non-proportional problems.

This is partly attributed to teachers' greater exposure to mechanical and algorithmic approaches to problem-solving from their formative stages, as well as the effect of the experimental contract that leads them to assume expectations and provide stereotyped responses. In contrast, individuals with limited education showed less attachment to these school dynamics, as they approached problems in a more flexible way and connected them to their daily experiences.

It is evident that the study has limitations. One of them was the size of the sample, so studies in this line of research should have larger samples, as well as design educational strategies that allow overcoming the illusion of linearity from early stages, promoting the discrimination of linear and non-linear situations through the resolution of contextualized and meaningful problems. On the other hand, these studies should delve deeper into the responses. Although some questions were asked when an interesting response from the participants was detected, an interview protocol was not designed to lead to a more in-depth understanding of their reasoning.

It's important to train teachers to recognize and address the illusion of linearity in their mathematics teaching, fostering more flexible and critical thinking among students. Future research with larger samples and more structured interview protocols is suggested to obtain a deeper understanding of participants' reasoning processes.

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Appendix

Test on the Illusion of Linearity

Name (if you agree to provide it) ____

Age_____ Maximum level of education____

Length of service(or job)____

For each of the following questions, take as much time as you consider necessary to solve them. Explain your answer in detail.

1) Pedro completes a 100-meter dash in 18 seconds. How much time will it take him to run 1000 meters?

2) If 5 candies cost $10, how much will 8 candies cost?

3) If a 10-meter ladder is needed to reach the roof of a house that is 8 meters high, what must be the length of a ladder to reach the roof of a building that is 80 meters high?

4) If a plant grows 10 centimeters in height in one day, how much will the plant grow in 30 days?

5) If a freshly washed sheet takes 2 hours to dry in the sun, how many hours will it take for 4 sheets to dry?

Test on the Illusion of Linearity

Name (if you agree to provide it) ____

Age_____ Maximum level of education_______________________________________

Length of service (or job)____

For each of the following questions, take as much time as you consider necessary to solve them. Explain your answer in detail.

1) A theater has a capacity for 1500 people. If it sells 900 tickets in 2 hours. How many tickets will it sell in 4 hours?
2) If 5 candies cost $10, how much will 8 candies cost?
3) If a ladder of 5 meters long is needed to reach the first branches of a tree that is 10 meters tall, what should be the length of a ladder to reach the window of a building that is 100 meters tall?
4) If a plant grows 10 centimeters in height in a day, how much will the plant grow in 30 days?
5) If a freshly washed sheet is put out in the sun and it takes 2 hours to dry, how many hours will it take for 4 sheets to dry?